Combinatorial Proofs

2.6.22:

(a) Give a combinatorial proof that if n is an odd integer, then the number of ways to select an even number of objects from a set of n objects is equal to the number of ways to select an odd number of objects.

(b) Give a combinatorial proof of Vandermonde's identity: For positive integers m and n and an integer r such that $0 \le r \le n+m$, $\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \binom{n}{2}\binom{m}{r-2} + \cdots + \binom{n}{r}\binom{m}{0}$.

(c) Prove that $\binom{2n}{n} + \binom{2n}{n+1} = \frac{1}{2}\binom{2n+2}{n+1}$.

(a) If n is an odd integer, and $k < n$ is even, then $n-k$ is odd. Likewise, if $k < n$ is odd, $n-k$ is even. Thus, every way of choosing an even number of objects selects an odd number of objects as well (those not chosen), and every way of choosing an odd number of objects selects an even number (likewise). Since every way of choosing an even number of objects selects an odd number and every way of choosing an odd number comes from an even number, the number of ways to do one is the same as the number of ways to do the other.

(b) Suppose we have $n+m$ objects, divided into two subsets, one of size n (the first n, perhaps), and one of size m (the last m). Then there are $\binom{n+m}{r}$ ways of selecting r objects from the full collection. However, we could also choose r objects by specifying how many come from the subset of size n, some number s, and then choosing $r-s$ from the subset of size m. There are $\binom{n}{s}$ ways of choosing from the first subset and $\binom{m}{r-s}$ from the second, for a total of $\binom{n}{s}\binom{m}{r-s}$. We could have any $0 \le s \le r$, so the total number of ways of choosing r objects is the sum over choices of s of such forms.

(c) We proceed in two stages. First, we prove

$$\binom{2n}{n} + \binom{2n}{n+1} = \binom{2n+1}{n+1}.$$

The left hand side counts the ways to choose either n or $n+1$ of $2n$ objects; we describe an alternate procedure to accomplish the same result. To the collection of $2n$ original objects, we add another, "null" object. From this collection of $2n+1$ objects, we select $n+1$. If the "null" object was selected, we have chosen n original objects; otherwise, we have chosen $n+1$ original objects. Any method for selecting n original objects corresponds to picking those objects and "null", and any method for selecting $n+1$ original objects works by simply not choosing "null". Thus, as the same outcomes are reached the same number of times, the number of ways to choose $n+1$ objects from $2n+1$ corresponds to the original task, and the identity is established.

Second, we prove

$$2\binom{2n+1}{n+1} = \binom{2n+2}{n+1}.$$

From $2n+2$ objects, distinguish one as "marked". If we select $n+1$ objects from the $2n+2$, either the "marked" object is in the $n+1$ chosen objects, or it is in the $n+1$ objects not chosen. Consider the task of selecting $n+1$ objects from the $2n+2$. Suppose we knew beforehand that the "marked" object will not be chosen; then the task is equivalent to choosing $n+1$ from the $2n+1$ "unmarked" objects. Suppose we knew beforehand that the "marked" object will be chosen; then the task is equivalent to eliminating the $n+1$ objects from the $2n+1$ "unmarked" objects that will not be chosen. Thus, the number of ways of selecting $n+1$ objects from $2n+2$ is $\binom{2n+1}{n+1} + \binom{2n+1}{n+1} = 2\binom{2n+1}{n+1}$. Together with the previous result, this shows that

$$\binom{2n}{n} + \binom{2n}{n+1} = \frac{1}{2}\binom{2n+2}{n+1}.$$

2.6.23: Give a combinatorial argument that $n^2 = 2\binom{n}{2} + n$.

Suppose we have n^2 objects we arrange in an $n \times n$ square, and assign row and column numbers from 1 to n to each object. To specify on of the n^2 objects, we must give the row and column numbers. The row number is either less than, equal to, or greater than the column number. Supposing we knew we were in the first case, we select two numbers from $\{1, \ldots, n\}$ and choose the smaller as the row number; there are $\binom{n}{2}$ ways of doing this. Supposing we knew we were in the third case, we similarly select two numbers from $\{1, \ldots, n\}$ and choose the larger as the row number; there are again $\binom{n}{2}$ ways of doing this. Finally, supposing we were in the second case, we simply choose one of the n numbers, and set it as both row and column coordinate. Every object falls into exactly one of these cases and is selected exactly once, so the total number of ways to proceed, $2\binom{n}{2} + n$, counts the n^2 objects.