Relations

3.1.3: Find the domain and range of the relation W on $\mathbb{R}$ given by $x\,W\,y$ if

(a) $y = 2x+1$

(b) $y = x^2+3$

(c) $y = \sqrt{x-1}$

(d) $y = \frac{1}{x^2}$

(a) The expression on the right is defined for all $x \in \mathbb{R}$, so the domain is $\mathbb{R}$, and we can solve for x, yielding $x = \frac{1}{2}(y-1)$, which is defined for all $y \in \mathbb{R}$. This tells us the inverse relation explicitly, and shows that the range is also $\mathbb{R}$.

(b) The expression on the right is defined for all $x \in \mathbb{R}$, so the domain is $\mathbb{R}$. We know that $x^2 \geq 0$ for all $x \in \mathbb{R}$, so the range is contained in $\{y \in \mathbb{R} : y \geq 3\}$. In fact, this is the range, as any such y is achieved by choosing $x = \sqrt{y-3}$ (or $x = -\sqrt{y-3}$); it is important to note that the condition $y \geq 3$ is necessary and sufficient to ensure this is defined in the reals.

(c) The expression on the right is defined for all $x \geq 1$, so the domain is $\{x \in \mathbb{R} : x \geq 1\}$. If $y \geq 0$, then $x\,W\,y$ if $x = y^2+1$ (and the non-negativity condition is crucial because we are taking the positive square root), so the range is $\{y \in \mathbb{R} : y \geq 0\}$.

(d) The expression on the right is defined for $x \in \mathbb{R}-\{0\}$, so $\mathbb{R}-\{0\}$ is the domain. Since $x^2 > 0$ for $x \neq 0$, the range is contained in $\{y \in \mathbb{R} : y > 0\}$; in fact this is the range, as setting $x = \frac{1}{\sqrt{y}}$ achieves y if $y > 0$.

3.1.6: Find the inverse of each relation. Express the inverse as the set of all pairs (x,y) subject to some condition.

(a) $R_1 = \{(x,y) \in \mathbb{R}\times\mathbb{R} : y = x\}$

(b) $R_2 = \{(x,y) \in \mathbb{R}\times\mathbb{R} : y = -5x+2\}$

(f) $R_6 = \{(x,y) \in \mathbb{R}\times\mathbb{R} : y < x+1\}$

(g) $R_7 = \{(x,y) \in \mathbb{R}\times\mathbb{R} : y > 3x-4\}$

(h) $R_8 = \left\{(x,y) \in \mathbb{R}\times\mathbb{R} : y = \frac{2x}{x-2}\right\}$

(a) Since $x\,R_1\,y$ iff $y = x$ iff $y\,R_1\,x$, the inverse relation here is $R_1^{-1} = \{(x,y) \in \mathbb{R}\times\mathbb{R} : y = x\}$ (that is, $R_1 = R_1^{-1}$). Technically, we've exchanged the roles of x and y (but equality is a symmetric relation).

(b) Here, $y\,R_2\,x$ iff $x = -5y+2$, or $y = \frac{1}{5}(2-x)$, so $R_2^{-1} = \left\{(x,y) \in \mathbb{R}\times\mathbb{R} : y = \frac{1}{5}(2-x)\right\}$. Note that we've again exchanged the roles of x and y.

(f) Here, $y\,R_6\,x$ iff $x < y+1$, or $y > x-1$, so $R_6^{-1} = \{(x,y) \in \mathbb{R}\times\mathbb{R} : y > x-1\}$.

(g) Here, $y\,R_7\,x$ iff $x > 3y-4$, or $y < \frac{1}{3}(x+4)$, so $R_7^{-1} = \left\{(x,y) \in \mathbb{R}\times\mathbb{R} : y < \frac{1}{3}(x+4)\right\}$.

(h) Here, $y\,R_8\,x$ iff $x = \frac{2y}{y-2}$, or $y \neq 2$ and $y = \frac{2x}{x-2}$. We note that attempting to solve $2 = \frac{2x}{x-2}$ for x fails: there is no x such that $y = 2 = \frac{2x}{x-2}$. Thus, we can write $R_8^{-1} = \left\{(x,y) \in \mathbb{R}\times\mathbb{R} : y = \frac{2x}{x-2}\right\}$. As an interesting fact, we can write $R_8 = R_8^{-1} = \{(x,y) \in \mathbb{R}\times\mathbb{R} : 2x - xy + 2y = 0 \wedge x \neq 2 \wedge y \neq 2\}$ to highlight the symmetry in the relation.

3.1.11: Let R be a relation from A to B and S be a relation from B to C.

(a) Prove that $\mathrm{Rng}(R^{-1}) = \mathrm{Dom}(R)$.

(b) Prove that $\mathrm{Dom}(S \circ R) \subseteq \mathrm{Dom}(R)$.

(a) If $y \in \mathrm{Rng}(R^{-1})$, that is equivalent to the existence of an x such that $x R^{-1} y$. This happens if and only if there is an x such that $y R x$, which is the definition of $y \in \mathrm{Dom}(R)$. Since the logical connectives here were biconditionals, we've shown equality (the other direction is automatic).

(b) Suppose $x \in \mathrm{Dom}(S \circ R)$. This means that there is some $y \in B$ and $z \in C$ such that $x R y$ and $y S z$. Ignoring the role of z, the fact that $x R y$ for some y shows that $x \in \mathrm{Dom}(R)$. Since x was arbitrary among $\mathrm{Dom}(S \circ R)$, this shows $\mathrm{Dom}(S \circ R) \subseteq \mathrm{Dom}(R)$.