3.1.3: Find the domain and range of the relation $W$ on $\mathbb{R}$ given by $x W y$ if

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<td>(a) $y = 2x + 1$</td>
<td>(c) $y = \sqrt{x - 1}$</td>
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<tr>
<td>(b) $y = x^2 + 3$</td>
<td>(d) $y = \frac{1}{x^2}$</td>
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(a) The expression on the right is defined for all $x \in \mathbb{R}$, so the domain is $\mathbb{R}$, and we can solve for $x$, yielding $x = \frac{1}{2} (y - 1)$, which is defined for all $y \in \mathbb{R}$. This tells us the inverse relation explicitly, and shows that the range is also $\mathbb{R}$.

(b) The expression on the right is defined for all $x \in \mathbb{R}$, so the domain is $\mathbb{R}$. We know that $x^2 \geq 0$ for all $x \in \mathbb{R}$, so the range is contained in $\{y \in \mathbb{R} : y \geq 3\}$. In fact, this is the range, as any such $y$ is achieved by choosing $x = \sqrt{y - 3}$ (or $x = -\sqrt{y - 3}$); it is important to note that the condition $y \geq 3$ is necessary and sufficient to ensure this is defined in the reals.

(c) The expression on the right is defined for all $x \geq 1$, so the domain is $\{x \in \mathbb{R} : x \geq 1\}$. If $y \geq 0$, then $x W y$ if $x = y^2 + 1$ (and the non-negativity condition is crucial because we are taking the positive square root), so the range is $\{y \in \mathbb{R} : y \geq 0\}$.

(d) The expression on the right is defined for $x \in \mathbb{R} - \{0\}$, so $\mathbb{R} - \{0\}$ is the domain. Since $x^2 > 0$ for $x \neq 0$, the range is contained in $\{y \in \mathbb{R} : y > 0\}$; in fact this is the range, as setting $x = \frac{1}{\sqrt{y}}$ achieves $y$ if $y > 0$.

3.1.6: Find the inverse of each relation. Express the inverse as the set of all pairs $(x, y)$ subject to some condition.

(a) $R_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x\}$

(b) $R_2 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = -5x + 2\}$

(f) $R_6 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y < x + 1\}$

(g) $R_7 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y > 3x - 4\}$

(h) $R_8 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \frac{2x}{x^2 - 2}\}$

(a) Since $xR_1y$ iff $y = x$ iff $yR_1x$, the inverse relation here is $R_1^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x\}$ (that is, $R_1 = R_1^{-1}$). Technically, we’ve exchanged the roles of $x$ and $y$ (but equality is a symmetric relation).

(b) Here, $yR_2x$ iff $x = -5y + 2$, or $y = \frac{1}{5} (2 - x)$, so $R_2^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \frac{1}{5} (2 - x)\}$. Note that we’ve again exchanged the roles of $x$ and $y$.

(f) Here, $yR_6x$ iff $x < y + 1$, or $y > x - 1$, so $R_6^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y > x - 1\}$.

(g) Here, $yR_7x$ iff $x > 3y - 4$, or $y < \frac{1}{3} (x + 4)$, so $R_7^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y < \frac{1}{3} (x + 4)\}$.

(h) Here, $yR_8x$ iff $x = \frac{2y}{y^2 - 2}$, or $y \neq 2$ and $y = \frac{2x}{x^2 - 2}$. We note that attempting to solve $2 = \frac{2x}{x^2 - 2}$ for $x$ fails: there is no $x$ such that $y = 2 = \frac{2x}{x^2 - 2}$. Thus, we can write $R_8^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \frac{2x}{x^2 - 2}\}$. As an interesting fact, we can write $R_8 = R_8^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : 2x - xy + 2y = 0 \land x \neq 2 \land y \neq 2\}$ to highlight the symmetry in the relation.
3.1.11: Let $R$ be a relation from $A$ to $B$ and $S$ be a relation from $B$ to $C$.

(a) Prove that $\text{Rng}(R^{-1}) = \text{Dom}(R)$.

(b) Prove that $\text{Dom}(S \circ R) \subseteq \text{Dom}(R)$.

(a) If $y \in \text{Rng}(R^{-1})$, that is equivalent to the existence of an $x$ such that $xR^{-1}y$. This happens if and only if there is an $x$ such that $yRx$, which is the definition of $y \in \text{Dom}(R)$. Since the logical connectives here were biconditionals, we’ve shown equality (the other direction is automatic).

(b) Suppose $x \in \text{Dom}(S \circ R)$. This means that there is some $y \in B$ and $z \in C$ such that $xRy$ and $ySz$. Ignoring the role of $z$, the fact that $xRy$ for some $y$ shows that $x \in \text{Dom}(R)$. Since $x$ was arbitrary among $\text{Dom}(S \circ R)$, this shows $\text{Dom}(S \circ R) \subseteq \text{Dom}(R)$. 