Equivalence Relations

3.2.2: Let $A = \{1, 2, 3\}$. List the ordered pairs, and draw the digraph of a relation on A with the given properties.

(a) not reflexive, not symmetric, and not transitive

(b) reflexive, not symmetric, and not transitive

(c) not reflexive, symmetric, and not transitive

(d) reflexive, symmetric, and not transitive

(e) not reflexive, not symmetric, and transitive

(a) The relation $\{(1,2),(2,3)\}$ is not reflexive because it does not contain $(1,1)$, not symmetric because it does not contain $(2,1)$, and not transitive because it does not contain $(1,3)$.

(b) The relation $\{(1,1),(2,2),(3,3),(1,2),(2,3)\}$ is not symmetric because it does not contain $(2,1)$ and not transitive because it does not contain $(1,3)$, but is reflexive.

(c) The relation $\{(1,2),(2,1)\}$ is not reflexive and not transitive because it does not contain $(1,1)$, but it is symmetric.

(d) The relation $\{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)\}$ is not transitive because it does not contain $(1,3)$, but it is reflexive and symmetric.

(e) The relation $\{(1,2),(2,3),(1,3)\}$ is not reflexive because it does not contain $(1,1)$ and not symmetric because it does not contain $(2,1)$, but it is transitive.

3.2.17: Prove that if R is a symmetric, transitive relation on A and the domain of R is A, then R is reflexive on A.

Let $a \in A$. Since $a \in \text{Dom}(R) = A$, we know there is some $b \in A$ such that $(a, b) \in R$. Since R is symmetric, we know that $(b, a) \in R$. Since R is transitive and (a, b) and (b, a) are in R, we know that $(a, a) \in R$. Since a was arbitrary, this shows that R is reflexive.