The Row Reduction Algorithm

The Punch Line: Given any linear system of equations, there is a procedure which finds a particularly simple equivalent system.

Warm-Up: For each of these matrices, determine if it is in Echelon Form, Reduced Echelon Form, or neither.

(a)
$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(c)
$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(d)
$$\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(e)
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(f)
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Neither

(b) Reduced Echelon Form

(c) Neither

(d) Echelon Form

(e) Reduced Echelon Form

(f) Reduced Echelon Form

Using the Algorithm: Five steps transform any matrix into a row-equivalent Reduced Echelon Form matrix:

1) Identify the pivot column. This will be the leftmost column with a nonzero entry.

2) Select a nonzero entry in that column to be the pivot for that column. If necessary, interchange rows to put it at the top of the matrix.

3) Eliminate all of the nonzero entries in the pivot column by using row replacement operations.

4) Repeat steps 1)-3) on all rows you haven't yet used.

5) Eliminate all nonzero entries above each pivot, and scale each nonzero row so its pivot is 1.

1. Transform each of these matrices into Reduced Echelon Form with the above procedure:

(a) $$\begin{bmatrix} 1 & 1 & 1 \\ -2 & 4 & -2 \\ -1 & 0 & 1 \end{bmatrix}$$

(b) $$\begin{bmatrix} 2 & 0 & 4 & 0 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 3 & 3 \end{bmatrix}$$

(c) $$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

(a) We already have a 1 in leading position in the first row, first column, so we identify the first column as the pivot column and select the 1 in the upper left to be the pivot. Then, we add twice Row 1 to Row 2, and add Row 1 to Row 3, to yield the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 6 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

Having eliminated all of the other nonzero entries in the pivot column, we repeat for the second pivot position. There are entries in the second column, so that will be our new pivot column. To save future work, I will select the 6 in the second row as our pivot—it would be equally valid to select the 1 in the third row and interchange Rows 2 and 3 to put it as the next pivot entry. With 6 as the pivot, though, we can add $\frac{-1}{6}$ times Row 2 to Row 3 to get the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, which is in Echelon Form.

In order to get Reduced Echelon Form, we apply step 5. First, we can multiply Row 2 by $\frac{1}{6}$ and Row 3 by $\frac{1}{2}$ to get the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then, we eliminate the entries above the pivots by subtracting both Row 2 and Row 3 from Row 1 to get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This is in Reduced Echelon Form, so we're done!

(b) $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Interpreting the Results: The Reduced Echelon Form of the augmented matrix of a linear system can be used to find all solutions (the *solution set*) of the system at once. To do this, we write out the system corresponding to the Reduced Echelon Form matrix, then solve for all of the variables in pivot positions (we can do this easily because each one only appears in a single equation). Any remaining variables are called *free variables*, and can take on any value in a solution.

2. Find the solution set of each of these linear systems:

(a)

$$\begin{aligned} 2x_1 + 4x_3 &= 0 \\ x_2 + x_3 &= 3 \\ x_1 + x_2 + 3x_3 &= 3 \end{aligned}$$

(b)

$$\begin{aligned} x + y &= 1 \\ 4y - 2x &= -2 \\ -x &= 1 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 + 2x_2 &= 1 \\ 2x_1 + 3x_2 &= 3 \end{aligned}$$

(a) The augmented matrix of this system is $\begin{bmatrix} 2 & 0 & 4 & 0 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 3 & 3 \end{bmatrix}$. From Part 1, we know that this has Reduced Echelon Form $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. This corresponds to the system of equations

$$\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 + x_3 &= 3 \\ 0 &= 0. \end{aligned}$$

Since x_1 and x_2 correspond to the pivot positions in the augmented matrix, we solve for them in terms of the free variable x_3: $x_1 = -2x_3$ and $x_2 = 3 - x_3$. Since x_3 is not in a pivot position, it is a free variable, which means that we can't pick a definite value for it. Instead, there is an infinite number of solutions to the system of equations, one for *every possible* value of x_3. In these solutions, x_1 and x_2 are defined by the equations we just calculated.

(b) The augmented matrix of this system is $\begin{bmatrix} 1 & 1 & 1 \\ -2 & 4 & -2 \\ -1 & 0 & 1 \end{bmatrix}$ (pay special attention to Row 2 there; it's important to keep the variables in the same order in all rows of the augmented matrix, so because we had x before y in Row 1, we need to do the same in Row 2, even though the equation is given to us the other way around). This has Reduced Echelon form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The last row corresponds to the equation $0 = 1$, which is never true. This means that the system can't have any solutions, so it is *inconsistent*.

(c) The augmented matrix of this system is $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 3 & 3 \end{bmatrix}$, which has Reduced Echelon Form $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. This corresponds to the system

$$\begin{aligned} x_1 &= 3 \\ x_2 &= -1 \\ 0 &= 0. \end{aligned}$$

The equation $0 = 0$ is always true, so we can read off the solution to the system $x_1 = 3$ and $x_2 = -1$.

Under the Hood: The Reduced Echelon Forms of any two equivalent systems are the same. Since equivalent systems have the same solution set (by definition!), it is in some sense the simplest system with that solution set. Thus, the Row Reduction Algorithm is a way to find the simplest description of the solution set of a linear system—that works every time!