Matrix-Vector Products

The Punch Line: We can use even more compact notation than vector equations by introducing matrices. This will allow us to study systems of linear equations by studying matrices.

Warm-Up: Write the following systems of linear equations as vector equations:

(a) The system with variables z_1 and z_2

$$\begin{aligned} z_1 + 2z_2 &= 6 \\ 2z_1 - 5z_2 &= 3. \end{aligned}$$

(b) The system with variables x, y, and z

$$\begin{aligned} x &= x_0 \\ y &= y_0 \\ z &= z_0. \end{aligned}$$

(c) The system with variables x_1, x_2, and x_3

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 - x_3 &= 0. \end{aligned}$$

(a) $z_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + z_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

(b) $x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$

(c) $x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

The Technique: The linear combination $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$ is represented by the matrix-vector product

$$A\vec{x} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

This means that to compute a matrix-vector product, we can just write it back out as a linear combination of the columns of the matrix. This means that matrix-vector products only work when there are precisely as many columns in the matrix as there are entries in the vector.

1 Compute the following matrix-vector products:

(a) $\begin{bmatrix} 1 & 2 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

(a) We write this as

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} &= 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 2 \\ -5 \end{bmatrix} \\ &= \begin{bmatrix} 4(1) + 1(2) \\ 4(2) + 1(-5) \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 3 \end{bmatrix}. \end{aligned}$$

(b) This is $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(c) This is $\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$.

(d) This is $\begin{bmatrix} 5 \\ 17 \end{bmatrix}$.

Applications: The matrix equation $A\vec{x} = \vec{b}$ can be rephrased as the assertion that $\vec{b}$ is in the span of the columns of A. This gives us a geometric interpretation of systems of linear equations when we write them in matrix form—an equation being true means a particular vector, $\vec{b}$, is in the span of the collection of vectors $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ that make up the matrix A. In this case, the vector $\vec{x}$ is the collection of weights in a linear combination that proves $\vec{b}$ is in the span of the columns of A.

2 If possible, find at least one solution to each of these matrix equations (if not, explain why it is impossible):

(a) $\begin{bmatrix} 1 & 2 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(a) We have seen that $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is a solution. To verify (and find any others), we write the augmented matrix $\begin{bmatrix} 1 & 2 & 6 \\ 2 & -5 & 3 \end{bmatrix}$. This has Reduced Echelon Form $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}$. From this, we can see that $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is the unique solution (and, if we hadn't already done the multiplication from the previous problem, we have derived it from just the equations).

(b) We start with the augmented matrix $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & -2 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$. This has REF $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, so we see the unique solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(c) The augmented matrix here is $\begin{bmatrix} 1 & 1 & b_1 \\ 1 & -1 & b_2 \\ 2 & 0 & b_3 \end{bmatrix}$. We work just with the left columns of the augmented matrix, and find that in REF, it looks like $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{bmatrix}$. This only works for some values that we could put into the $*$s, but not in general. This means that this matrix equation is inconsistent for (most) $\vec{b}$ (and, therefore, that the columns of the matrix do not span $\mathbb{R}^3$).

(d) Here, the REF of the augmented matrix is $\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix}$. We have a free variable in this, so there are infinitely many solutions. We can choose a value for x_3 to get a particular solution—choosing $x_3 = 0$ gives the solution $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, while choosing $x_3 = 1$ yields $\begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$. In fact, the set of all solutions can be represented as $\vec{x} = t \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$, which forms a line (more on this in the next section of the book).

Under the Hood: Given any vector $\vec{b}$, the equation $A\vec{x} = \vec{b}$ means that $\vec{b}$ is in the span of the columns of A. This means that the span of the columns of A is related to the set of all possible matrix equations that could be solved with $A\vec{x}$ as the left hand side—there's one for each $\vec{b}$ in the span!