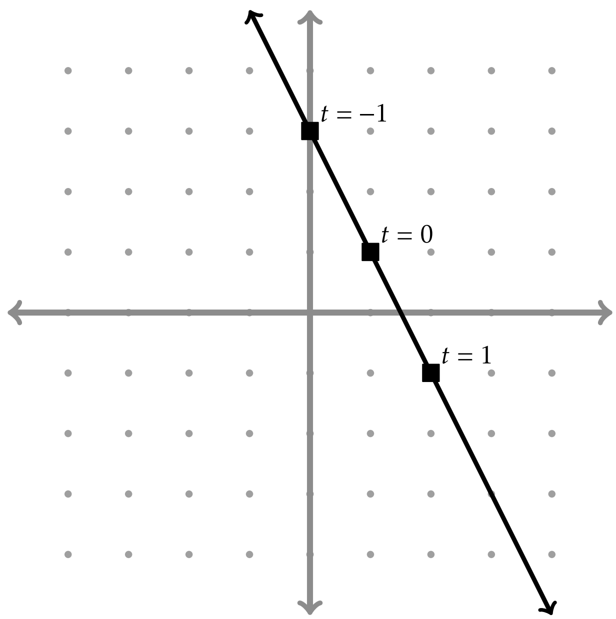
Solution Sets of Linear Systems

The Punch Line: There is a geometric interpretation to the solution sets of systems 0f linear equations, which allows us to explicitly describe them with *parametric equations*.

Warm-Up: Draw the line in $\mathbb{R}^2$ defined by $y = 3 - 2x$.



Verify that $x(t) = 1 + t$ and $y(t) = 1 - 2t$ satisfy the equation $y(t) = 3 - 2x(t)$ for all t, and plot $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ for $t = -1$, 0, and 1.

We compute $2x(t) + y(t) = 2(1 + t) + (1 - 2t) = 2 + t + 1 - 2t = 3$. Thus, $y(t) = 3 - 2x(t)$.

Homogeneous Equations: A matrix equation of the form $A\vec{x} = \vec{0}$ is called *homogeneous*. It always has the solution $\vec{x} = \vec{0}$, which is called the *trivial solution*. Any other solution is called a *nontrivial solution*; nontrivial solutions arise precisely when there is at least one free variable in the equation.

If there are m free variables in the homogeneous equation, the solution set can be expressed as the span of m vectors:

$$\vec{x} = s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_m\vec{v}_m.$$

This is called a *parametric equation* or a *parametric vector form* of the solution. A common parametric vector form uses the free variables as the parameters s_1 through s_m.

1 Find a parametric vector form for the solution set of the equation $A\vec{x} = \vec{0}$ for the following matrices A:

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ 4 & -4 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & -2 & 0 \\ -2 & 0 & 4 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

(a) We compute the REF of $\begin{bmatrix} A & \vec{0} \end{bmatrix}$, finding it to be $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This means x_2 is a free variable, so solve for the relation $x_1 = -2x_2$. Since we know $x_2 = x_2$ (and can't say anything more, because x_2 is free), we can express our solution in the parametric form

$$\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}.$$

(b) Here our REF is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, so the only solution is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. We could write this as the parameterized version $\vec{x} = s \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$, but it's simplest to just leave it as $\vec{0}$ (which is still an explicit solution to the equation).

(c) The REF is $\begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. This means x_2, x_3, and x_4 are all free variables. We express $x_1 = 2x_3$. With this, we can write our parametric solution as

$$\vec{x} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

(d) The REF here is again $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, so the only solution is $\vec{0}$.

Nonhomogeneous Equations: A matrix equation of the form $A\vec{x} = \vec{b}$ where $\vec{b} \neq \vec{0}$ is called *nonhomogeneous*. As we've seen, a nonhomogeneous system may be inconsistent and fail to have solutions. If it does have a solution, though, we can find a parametric form for them as well as in the homogeneous case. Here, we express the solutions as $\vec{x} = \vec{p} + \vec{v}_h$, where $\vec{p}$ is some particular solution to the nonhomogeneous system (which we can get by picking simple values for the parameters, such as taking all free variables to be zero), and $\vec{v}_h$ is a parametric form for the solution to the *homogeneous* equation $A\vec{v}_h = \vec{0}$.

2 If possible, find a parametric vector form for the solution set of the nonhomogeneous equation $A\vec{x} = \vec{b}$ for the following matrices A and vectors $\vec{b}$ (otherwise explain why it is impossible):

(a) $\begin{bmatrix} 1 & 2 \end{bmatrix}; \begin{bmatrix} 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}; \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}; \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(a) Since the augmented matrix of this system, $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, is already in REF, we can immediately solve for the pivot variable x_1 in terms of the free variable x_2 and constants. We get $x_1 = 3 - x_2$. To get a parametric vector form for the solution, we first choose the value $x_2 = 0$ to see that the vector $\vec{p} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ is *a* solution of the nonhomogeneous equation. To get the rest, we look at the homogeneous system with the same matrix A (this amounts to looking at the REF with the last column set to all zeros, or seeing how changing the free variables change the pivot variables). In this case, any vector of the form $s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a solution to the homogeneous equation (increasing x_2 decreases x_1 by the same amount), so our final parametric vector form of the answer is

$$\vec{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(b) The REF of this system is $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{-1}{2} \end{bmatrix}$. There are no pivot variables, so the only solution we get is

$$\vec{x} = \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2} \end{bmatrix}.$$

This is in parametric form, although there are no parameters.

(c) The REF here is $\begin{bmatrix} 1 & 0 & 0 & \frac{5}{2} \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -4 \end{bmatrix}$. This again has a unique solution, $\vec{x} = \begin{bmatrix} \frac{5}{2} \\ -2 \\ -4 \end{bmatrix}$.

(d) The REF here is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Since the last column is a pivot, the system is inconsistent, so the solution set is empty—there are no vectors $\vec{x}$ which solve this equation.

Under the Hood: Why do the solution sets to nonhomogeneous solutions have a "homogeneous part"? Imagine we are given two vectors, $\vec{x}_1$ and $\vec{x}_2$, and we're assured that $A\vec{x}_1 = \vec{b}$ and $A\vec{x}_2 = \vec{b}$. That is, we have two solutions to the nonhomogeneous equation. We can take the difference between these two equations to see that $A\vec{x}_1 - A\vec{x}_2 = \vec{b} - \vec{b}$. A property of matrix-vector multiplication lets us write the left-hand side as $A(\vec{x}_1 - \vec{x}_2)$, while the right-hand side is clearly $\vec{0}$, so we're left with the equation $A(\vec{x}_1 - \vec{x}_2) = \vec{0}$. That is, we've just shown the *difference* between two solutions to the nonhomogeneous equation is always a solution to the homogeneous equation with the same matrix!