# Linear Transformations

**The Punch Line:** Matrix multiplication defines a special kind of function, known as a *linear transformation*.

**Warm-Up:** What do each of these situations mean (geometrically, algebraically, in an application, and/or otherwise)?

(a) The product of the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

(b) The vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(c) The equation $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has a solution.

(d) The set of vectors $\vec{x}$ such that the matrix equation $A\vec{x} = \vec{b}$ is satisfied forms a plane in $\mathbb{R}^3$.

(e) The set of vectors $\vec{b}$ such that the matrix equation $A\vec{x} = \vec{b}$ is satisfied forms a line in $\mathbb{R}^2$.

(f) For two particular vectors $\vec{x}$ and $\vec{b}$, and a matrix $A$, the matrix equation $A\vec{x} = \vec{b}$ is satisfied.

You might have more answers (and I would love to talk about them in office hours!), but here are some helpful ones:

(a) The matrix rotates the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by 90° (or $\frac{\pi}{2}$ radians) counterclockwise to the vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. In fact, the matrix rotates *any* vector by that angle, as you can check.

(b) There is a way to take a linear combination of the three vectors that yields the all-ones vector.

(c) There is a vector $\vec{x}$ that the matrix sends to (or transforms into) the all-ones vector.

(d) There are multiple linear combinations of the columns of $A$ that yield $\vec{b}$, and $A$ sends (infinitely) many vectors in $\mathbb{R}^3$ to $\vec{b}$.

(e) The span of the columns of $A$ is a line, and $A$ transforms any vector it multiplies into a multiple of some particular vector.

(f) The matrix $A$ transforms the vector $\vec{x}$ into $\vec{b}$.

**What They Are:** A *linear transformation* is a mapping $T$ that obeys two rules:

(a) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}$ and $\vec{v}$ in its domain,

(b) $T(c\vec{u}) = cT(\vec{u})$ for all scalars $c$ and $\vec{u}$ in its domain.

These rules lead to the rule $T(c\vec{u}+d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$ for $c, d$ scalars and $\vec{u}, \vec{v}$ in the domain of $T$, and in fact $T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_nT(\vec{v}_n)$. That is, the transformation of a linear combination of vectors is a linear combination of the transformations of the vectors (with the same coefficients).

**1** Are each of these operations linear transformations? Why or why not?

(a) $T(\vec{x}) = 4\vec{x}$

(b) $T(\vec{x}) = A\vec{x}$ for some matrix $A$ (with the right number of columns)

(c) $T(\vec{x}) = \vec{0}$

(d) $T(\vec{x}) = \vec{b}$ for some nonzero $\vec{b}$

(e) $T(\vec{x}) = \vec{x}+\vec{b}$ for some nonzero $\vec{b}$

(f) $T(\vec{x})$ takes a vector in $\mathbb{R}^2$ and rotates it by 45° ($\frac{\pi}{4}$ radians) counter-clockwise in the plane

(a) Yes, because $4(\vec{u}+\vec{v}) = 4\vec{u}+4\vec{v}$ by the distributive property, and $4(c\vec{u}) = 4c\vec{u} = c(4\vec{u})$ by the associative and commutative properties of scalar multiplication.

(b) Yes, the two properties of linear transformations are properties of matrix multiplication.

(c) Yes, because $T(\vec{u}+\vec{v}) = \vec{0} = \vec{0}+\vec{0} = T(\vec{u}) + T(\vec{v})$ and $T(c\vec{u}) = \vec{0} = c\vec{0} = cT(\vec{u})$. Note that we can find a matrix $O$ (all of whose entries are zero) such that $O\vec{x} = \vec{0}$.

(d) No, because $T(c\vec{u}) = \vec{b}$, and $cT(\vec{u}) = c\vec{b}$, but $\vec{b} \neq c\vec{b}$ if $c \neq 1$ and $\vec{b} \neq \vec{0}$.

(e) No, because $T(\vec{u}+\vec{v}) = (\vec{u}+\vec{v})+\vec{b} = \vec{u}+\vec{v}+\vec{b}$, but $T(\vec{u}) + T(\vec{v}) = (\vec{u}+\vec{b}) + (\vec{v}+\vec{b}) = \vec{u}+\vec{v}+2\vec{b}$, which is different for $\vec{b} \neq \vec{0}$.

(f) Yes. It's pretty clear the $T(c\vec{x}) = cT(\vec{x})$, because rotating a vector doesn't change its length, so if the input was a multiple of $\vec{x}$, the output will be that same multiple of $T(\vec{x})$. It's probably easiest to convince yourself that the vector addition property works with a sketch, but the gist is that rotating both vectors by the same amount doesn't change the *relative* angle between them, so laying them tail-to-head after the rotation looks essentially the same except for the initial angle. As it turns out, there's a matrix that accomplishes this linear transformation as well:

$$T(\vec{x}) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \vec{x}.$$

It's not necessary to find this, but it does prove it's linear (by part (b)), and it's suggestive of things that will happen further along in the course...

**What They Do:** Linear transformations convert between two different spaces, such as $\mathbb{R}^n$ and $\mathbb{R}^m$. If $n = m$, then we can also think of them moving around the vectors inside $\mathbb{R}^n$ (e.g., by rotation or stretching).

2 What do the linear transformations corresponding to multiplication by these matrices do, geometrically? (Try applying the matrix to a vector composed of variables, then examining the result, or multiplying by a few simple vectors and sketching what happens.)

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

(a) This matrix does not change vectors it multiplies against.

(b) This matrix switches the coordinates of vectors it multiplies against, which reflects them about the line $y = x$.

(c) This rotates the $x$ and $y$ components by 90° (or $\frac{\pi}{2}$ radians), while leaving $z$ alone.

(d) This "projects" a vector onto the $z$ axis (it gives the vector that matches the input in height, but doesn't have any $x$ or $y$ components).

(e) This doubles the length of the vector.

(f) This quadruples the $y$ coordinate while leaving $x$ unchanged (this is sometimes called a shear transformation).