## The Inverse of a Matrix

**The Punch Line:** Undoing a linear transformation given by a matrix corresponds to a particular matrix operation known as *inverse*.

<b>Warm-Up:</b> Are the following vector operations reversible/invertible?		
(a) $T(\vec{x}) = 4\vec{x}$	(d) $T(\vec{x}) = \vec{x} + \vec{b}$	
(b) $T(\vec{x})$ is counterclockwise rotation in the plane by 45° ( $\frac{\pi}{4}$ radians)	(e) $T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}$	
(c) $T(\vec{x}) = \vec{0}$	(f) $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}$	

- (a) Yes, with inverse  $T^{-1}(\vec{x}) = \frac{1}{4}\vec{x}$ .
- (b) Yes, with an inverse given by clockwise 45° rotation.
- (c) No, because we can't tell what the input was if everything goes to the same place.
- (d) Yes, with inverse  $T^{-1}(\vec{x}) = \vec{x} \vec{b}$ ; note that this is not a linear transformation unless  $\vec{b} = \vec{0}$ .
- (e) Yes, by multiplying by the same matrix again (switching components twice puts them back where they began).
- (f) No, because  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$  are both sent to  $\vec{0}$ , so you can't tell which you started with by looking at the result.

**The Inverse:** The *inverse* of an  $n \times n$  matrix A is another matrix B that satisfies the two matrix equations  $AB = I_n$  and  $BA = I_n$ , where the *identity matrix*  $I_n$  has ones on the diagonal and zeroes everywhere else. We use the notation  $A^{-1}$  to refer to such a B (which, if it exists, is unique).

We can find the inverse of a matrix by applying row operations to the augmented matrix  $\begin{bmatrix} A & I_n \end{bmatrix}$  (which is augmented with the *n* columns of the identity matrix, rather than a single vector). If the left part of the augmented matrix can be transformed by row operations to  $I_n$ , then the right part will be transformed by those row operations to  $A^{-1}$ . If the system is inconsistent, the matrix *A* is not invertible (and we may call it *singular*).

1 Find the inverse of these matrices (you may want to check your results by multiplying the result with the original matrix):

(a) 
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
(c)  $\begin{bmatrix} 3 & -1 \\ 7 & -2 \end{bmatrix}$ (b)  $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ (d)  $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ 

- (a) We find the REF of the augmented matrix  $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ , which is  $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ , so the inverse of the matrix is  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ . Indeed,  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ . Compare to the argument in part e) of the Warm-Up.
- (b) This is  $\begin{bmatrix} \frac{1}{2} & 0\\ 0 & \frac{1}{3} \end{bmatrix}$ . (c) This is  $\begin{bmatrix} -2 & 1\\ -7 & 3 \end{bmatrix}$ .
- (d) This is  $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$

**Relevance to Matrix Equations:** The inverse of a matrix allows you to "reverse engineer" a matrix equation, in the sense that if  $A\vec{x} = \vec{b}$  and A is invertible, then  $\vec{x} = A^{-1}\vec{b}$  is a solution to the original equation. In fact, it is the unique solution to the equation!

2 Use the inverses computed previously t	to solve these matrix e	equations:
(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$	(c) $\begin{bmatrix} 3\\7 \end{bmatrix}$	$\begin{bmatrix} -1 \\ -2 \end{bmatrix} \vec{x} = \begin{bmatrix} a \\ a+1 \end{bmatrix}$
(b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$	(d) $\begin{bmatrix} 0\\1\\1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

- (a) We use the inverse calculated previously to get  $\begin{bmatrix} -1\\1 \end{bmatrix}$ .
- (b) Here we get  $\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$ .
- (c) We use the matrix as follows:  $\begin{bmatrix} -2 & 1 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} a \\ a+1 \end{bmatrix} = \begin{bmatrix} 1-a \\ 3-4a \end{bmatrix}$ .
- (d) Similarly,  $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{-a+b+c}{2} \\ \frac{a-b+c}{2} \\ \frac{a+b-c}{2} \\ \frac{a+b-c}{2} \end{bmatrix}.$

Computing the inverse of a matrix reveals the structure of how to invert the linear transformation it represents. As the book notes, it can be faster to simply perform row operations to find a solution to any particular matrix equation. However, looking at the inverse matrix can give a more geometric idea of what undoing some particular operation is—to undo a rotation and shear requiring a different shear and rotation in the opposite direction, for example.