Math 4A
Worksheet 2.8

Subspaces of $\mathbb{R}^n$

The Punch Line: Some parts of $\mathbb{R}^n$ behave exactly like copies of $\mathbb{R}^m$ (where m is smaller than n) that are sitting inside of the larger space.

Warm-Up

(a) In $\mathbb{R}^3$, if you add two vectors in the $y = 0$ plane, is the result guaranteed to be in the $y = 0$ plane?

(b) Is the answer the same or different for the $y = 1$ plane?

(c) In $\mathbb{R}^2$ if you take two vectors with x component greater than 1 and add them, is the result guaranteed to have an x component greater than 1?

(d) In $\mathbb{R}^2$, if you have a vector with x component greater than 1 and take a scalar multiple of it, is the result guaranteed to have an x component greater than 1?

(e) In $\mathbb{R}^2$, if you have two vectors that each lie on one of the axes, is their sum guaranteed to lie on an axis?

(f) In $\mathbb{R}^2$, if a vector lies on one of the axes and you take a scalar multiple of it, is the result guaranteed to be on one of the axes?

The Definition: A *subspace* of $\mathbb{R}^n$ is a subset[1] H that satisfies the following three properties:

i) H contains the vector $\vec{0}$

ii) If the vectors $\vec{u}$ and $\vec{v}$ are both in H, then so is $\vec{u}+\vec{v}$

iii) If the vector $\vec{u}$ is in H, then for any real number c the vector $c\vec{u}$ is in H

If we want to test if a subset H is a subspace, we just have to see if these properties hold for it.

1 Are these things subspaces?

(a) The subset $\{\vec{0}\}$ in any $\mathbb{R}^n$

(b) The $y = 0$ plane in $\mathbb{R}^3$

(c) The $y = 1$ plane in $\mathbb{R}^3$

(d) The vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ with $x \geq 1$

(e) The axes in $\mathbb{R}^2$

(f) The set of solutions to the matrix equation $A\vec{x} = \vec{0}$

(g) The set of solutions to the matrix equation $A\vec{x} = \vec{b}$ (where $\vec{b} \neq \vec{0}$)

(h) The span of the columns of the matrix A (for any matrix; for concreteness, feel free to think about 3×3 matrices in particular, although it is true for $m \times n$ matrices for any m and n)

[1] A *subset* of $\mathbb{R}^n$ is just some collection of vectors in $\mathbb{R}^n$.

A Basis: A *basis* for a subspace is a linearly independent set whose span is precisely that subspace. To check if a collection of vectors is a basis for a subspace H, we can put the vectors as the columns of a matrix B. Then the requirement that it is linearly independent is satisfied precisely if every *column* is a pivot column (equivalently, there are no free variables), and the requirement that the span is H is satisfied if the equation $B\vec{x} = \vec{b}$ has a solution precisely when $\vec{b} \in H$. In the special case that H is all of $\mathbb{R}^n$, these conditions are equivalent to B being invertible.

2 Are the following sets of vectors bases for the specified subspaces? (You may assume that it is indeed a subspace.)

(a) The set $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ for the subspace $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$

(b) The set $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}$ for the "subspace" $\mathbb{R}^2$

(c) The set $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ for the subspace Span $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$

(d) The set $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}-1\\2\\-1\end{bmatrix}\right\}$ for the subspace of $\mathbb{R}^3$ consisting of all vectors whose components sum to zero.

(e) The set $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$ for the subspace Span $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$

What's special about a subspace? It "looks like" $\mathbb{R}^m$ living inside $\mathbb{R}^n$. Eventually, we want to capitalize on this to break complicated descriptions into simpler ones. For example, we might be excited to discover that for a part of $\mathbb{R}^{37}$ that looks like $\mathbb{R}^2$, a particularly nasty linear transformation works just like rotation (even if it's hard to describe elsewhere). Subspaces are precisely the parts of $\mathbb{R}^n$ that work nicely with things like linear equations and transformations.