Determinants!

The Punch Line: We can compute a value from the entries of a matrix to get yet *another* way of characterizing invertible matrices. ***SPOILER ALERT***: The determinant will also give us a variety of other useful pieces of information in understanding a matrix and its associated linear transformation!

Warm-Up: Are these matrices invertible? Are there conditions that make them so or not so depending on certain values? Try to answer without reducing them to REF (and in general, with as few computations as possible).

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

(d) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & -1 \\ 1 & 1 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix}$

The Definition: We define the *determinant* of a matrix in general to be

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}).$$

In this definition, we're moving along the first row, taking (-1) to be one power higher than the column we're in (this means take a positive value for odd columns and a negative one for even columns), multiplying by the entry we find, then taking the determinant of the smaller matrix obtained by ignoring the top row and the column we're in. This involves the determinant of smaller matrices, so if we drill down enough layers, we'll get back to 2×2 matrices, where we can just use the formula $ad - bc$ from earlier.

1 Find the determinants for each of these matrices:

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 1 & -1 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ -1 & -2 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix}$

Cofactor Expansion: We can actually expand along *any* row or column. In that case, the (-1) has exponent $(-1)^{i+j}$ (where i marks the row and j the column we're in), the matrix entry is a_{ij}, and the subdeterminant is $\det(A_{ij})$. The goal here is to find the simplest row or column to move along to minimize the amount of computation. Mostly, this means finding the row or column with the most zeros, and the "nicest" (e.g., smallest) nonzero entries.

2 Try to compute the determinants of the following matrices by computing as few subdeterminants as possible.

(a) $\begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & -1 \\ -2 & 0 & 3 \end{bmatrix}$ (min is 1 subdeterminant)

(b) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 4 \\ 1 & 0 & -1 \end{bmatrix}$ (min is 2 subdeterminants)

(c) $\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ (min is 3 subdeterminants if you use a result from problem 1, and 4 otherwise)

(d) $\begin{bmatrix} 2 & -1 & 0 & 3 \\ 7 & -2 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ -3 & -2 & 0 & 1 \end{bmatrix}$ (min is 2 subdeterminants)