Determinants II: Return of Row Operations

The Punch Line: We can use row operations to calculate determinants if we're careful.

The Process: Our three row operations—interchange, scaling, and replacement with a sum—have predictable effects on the determinant. By tracking the operations we use to get a matrix that is easy to compute a determinant for—generally a matrix in echelon form (which is also in triangular form)—we can avoid most of the work involved. In particular, interchanging two rows multiplies the determinant by -1, scaling a row by k scales the determinant by k, and replacing a row with its sum with a multiple of a *different* row does not change the determinant. In practice, we mostly want to interchange rows and use scaled sums to get to Echelon Form (not necessarily reduced!), then multiply the diagonal entries.

1 Use row operations to help compute these determinants:

(a) $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 1 & -1 \end{vmatrix}$

(c) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix}$

(d) $\begin{vmatrix} 1 & 2 & 3 & 0 & 4 \\ 12 & \sqrt{\pi} & e^{e^e} & 1 & \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{2}}}} \\ 0 & 0 & 2 & 0 & 6 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{2}{3} \\ 0 & \frac{1}{2} & 0 & 0 & 1 \end{vmatrix}$

(a) A pair of interchanges (top and bottom, followed by the middle pair) yields the identity (with determinant 1), so the determinant here is $(-1)^2 \det(I) = 1$.

(b) Putting this in Echelon Form (*not* reduced) gives $\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & -3 \end{bmatrix}$, with no interchanges necessary, so the determinant is $(1)(1)(2)(-3) = -6$.

(c) Here an Echelon Form with no interchanges gives $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$, so the determinant is $(1)(-2)(-2)(4) = 16$.

(d) For this one, we probably want to start with a cofactor expansion down the fourth column, getting

$$\begin{vmatrix} 1 & 2 & 3 & 0 & 4 \\ 12 & \sqrt{\pi} & e^{e^e} & 1 & \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{2}}}} \\ 0 & 0 & 2 & 0 & 6 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{2}{3} \\ 0 & \frac{1}{2} & 0 & 0 & 1 \end{vmatrix} = (-1)^{4+2} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 6 \\ 0 & \frac{1}{2} & 0 & \frac{2}{3} \\ 0 & \frac{1}{2} & 0 & 1 \end{vmatrix}.$$

We can then interchange the second and third rows (at the cost of a negative sign) and subtract the new second row from the fourth (without changing the determinant to see that

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 6 \\ 0 & \frac{1}{2} & 0 & \frac{2}{3} \\ 0 & \frac{1}{2} & 0 & 1 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & \frac{1}{2} & 0 & \frac{2}{3} \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & \frac{1}{3} \end{vmatrix} = -\frac{1}{3}.$$

Column Operations and Other Properties: Since $\det(A) = \det(A^T)$ (which takes a bit of argument to show), we can also do column operations analogous to the row operations, with the same effect on the determinant. Interspersing them can be helpful. Another useful property is that $\det(AB) = \det(A)\det(B)$ (although $\det(A+B)$ is often not $\det(A)+\det(B)$).

2 Find expressions for the following determinants (and justify them):

(a) $\det\left(A^2\right)$

(b) $\det(A^n)$

(c) $\det(BA)$

(d) $\det\left(A^{-1}\right)$

(e) $\det(kA)$ (where k is some real number)

(f) $\begin{vmatrix} A & O \\ O & B \end{vmatrix}$

In the last problem, A and B are standing for the entries of matrices A and B filling out those portions of the matrix, and O stands for zeros in those entries (so if A is $n \times n$ and B is $m \times m$, this matrix is $(n+m)\times(n+m)$. This is something of a challenge problem—I expect it's more abstract than most problems you'll be given.

(a) Since $\det(AB) = \det(A)\det(B)$, we have $\det\left(A^2\right) = \det(AA) = \det(A)\det(A) = \det(A)^2$.

(b) Repeating the above argument gives that $\det(A^n) = \det(A)^n$.

(c) We know that $\det(BA) = \det(B)\det(A)$. Since the determinant of a real matrix is just a real number, though, this is $\det(A)\det(B)$ (the determinants commute even if the matrices do not!).

(d) Since $\det(I) = 1$, and $I = AA^{-1}$, we get $\det\left(AA^{-1}\right) = \det(A)\det\left(A^{-1}\right) = 1$, which we can solve as $\det\left(A^{-1}\right) = \det(A)^{-1}$, as $\det(A) \neq 0$ so long as it's invertible.

(e) Since kA has *each* row multiplied by k, we see that $\det(kA) = k^n \det(A)$ (assuming we're in $\mathbb{R}^n$).

(f) If we use interchanges and add rows to each other to put this in echelon form, we see we get the pivot values of A along the upper part of the diagonal and of B along the lower part, so this determinant is $\det(A)\det(B)$.