Determinants II: Return of Row Operations

The Punch Line: We can use row operations to calculate determinants if we're careful.

The Process: Our three row operations—interchange, scaling, and replacement with a sum—have predictable effects on the determinant. By tracking the operations we use to get a matrix that is easy to compute a determinant for—generally a matrix in echelon form (which is also in triangular form)—we can avoid most of the work involved. In particular, interchanging two rows multiplies the determinant by -1, scaling a row by k scales the determinant by k, and replacing a row with its sum with a multiple of a *different* row does not change the determinant. In practice, we mostly want to interchange rows and use scaled sums to get to Echelon Form (not necessarily reduced!), then multiply the diagonal entries.

1 Use row operations to help compute these determinants:

(a) $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 1 & -1 \end{vmatrix}$

(c) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix}$

(d) $\begin{vmatrix} 1 & 2 & 3 & 0 & 4 \\ 12 & \sqrt{\pi} & e^{e^e} & 1 & \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{2}}}} \\ 0 & 0 & 2 & 0 & 6 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{2}{3} \\ 0 & \frac{1}{2} & 0 & 0 & 1 \end{vmatrix}$

Column Operations and Other Properties: Since $\det(A) = \det(A^T)$ (which takes a bit of argument to show), we can also do column operations analogous to the row operations, with the same effect on the determinant. Interspersing them can be helpful. Another useful property is that $\det(AB) = \det(A)\det(B)$ (although $\det(A+B)$ is often not $\det(A)+\det(B)$).

2 Find expressions for the following determinants (and justify them):

(a) $\det\left(A^2\right)$

(b) $\det(A^n)$

(c) $\det(BA)$

(d) $\det\left(A^{-1}\right)$

(e) $\det(kA)$ (where k is some real number)

(f) $\begin{vmatrix} A & O \\ O & B \end{vmatrix}$

In the last problem, A and B are standing for the entries of matrices A and B filling out those portions of the matrix, and O stands for zeros in those entries (so if A is $n\times n$ and B is $m\times m$, this matrix is $(n+m)\times(n+m)$. This is something of a challenge problem—I expect it's more abstract than most problems you'll be given.