Math 4A
Key 4.1

Vector Spaces

The Punch Line: The same ideas we've been using translate to work on more abstract vector spaces, which describe many things which occur in "nature" (at least, in the mathematics we use to describe nature).

The Rules: A *vector space* is a set of objects V that satisfy these 10 axioms:

1. $\vec{x}+\vec{y} \in V$ (we say V is closed under addition)
2. $\vec{x}+\vec{y} = \vec{y}+\vec{x}$ (addition is commutative)
3. $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$ (addition is associative)
4. There is a $\vec{0}$ with the property that $\vec{x}+\vec{0} = \vec{x} = \vec{0}+\vec{x}$
5. There is a $\overrightarrow{-x}$ for every $\vec{x}$ so $\vec{x}+\overrightarrow{-x} = \vec{0}$
6. $c\vec{x} \in V$ (V is closed under scalar multiplication)
7. $c(\vec{x}+\vec{y}) = c\vec{x}+c\vec{y}$ (left distributivity)
8. $(c+d)\vec{x} = c\vec{x}+d\vec{x}$ (right distributivity)
9. $c(d\vec{x}) = (cd)\vec{x}$ (scaling is associative)
10. $1\vec{x} = \vec{x}$ (multiplicative identity)

1 Are these things vector spaces?

(a) The subset $\left\{\vec{0}\right\}$ in any $\mathbb{R}^n$

(b) $\mathbb{R}^2$ but scalar multiplication $c\vec{x}$ is defined as $c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x/c \\ y/c \end{bmatrix}$.

(c) $\mathbb{R}^2$ but addition is defined as $\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} y+w \\ x+z \end{bmatrix}$

(d) All functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$

(e) The set of all functions of the form $f(\theta) = a\sin(\theta+\phi)$

(f) The set of vectors $\vec{b}$ such that $\vec{b} = A\vec{x}$ (where A is fixed)

(a) Yes—adding only the zero vector and scaling the zero vector don't do anything to it, and the operations work like in $\mathbb{R}^n$.

(b) No—the only axiom which fails is right distributivity, because $\frac{1}{c+d} \neq \frac{1}{c} + \frac{1}{d}$ in general.

(c) No—this addition is not associative, but more interestingly there is no zero element (adding the zero vector switches the components).

(d) Yes—adding two functions which vanish at zero won't make a nonzero entry there, nor will scaling it, and the operations being well-behaved can be checked.

(e) Yes—the sum of sines is another sine with a different amplitude and phase (you may need to refresh yourself on some trig identities for this).

(f) Yes—if $\vec{b} = A\vec{x}$ and $\vec{b}' = A\vec{x}'$, then $\vec{b}+\vec{b}' = A\vec{x}+A\vec{x}' = A(\vec{x}+\vec{x}')$, and $c\vec{b} = cA\vec{x} = A(c\vec{x})$, and the other axioms are properties of $\mathbb{R}^n$.

Subspaces: A subset U of a vector space V is a *subspace* if it contains $\vec{0}$ and is closed under addition and scaling. A subspace is a vector space in its own right.

2 Are these subsets subspaces?

(a) The vectors in $\mathbb{R}^3$ whose entries sum to zero

(b) The vectors in $\mathbb{R}^2$ which lie on one of the axes

(c) The vectors in $\mathbb{R}^3$ that are mapped to zero by matrices A and B

(d) The functions of the form $f(\theta) = A\sin(\theta + \phi)$ with ϕ rational

(e) The functions of the form $f(\theta) = A\sin(\theta + \phi)$ with ϕ irrational

(f) The functions of the form $f(\theta) = A\sin(\theta + \phi)$ with A rational

(a) Yes—$\vec{0}$ is obviously in, and its quick to check that the sum of the entries of the sum of the two vectors is the sum of the sums of their entries, which is zero, and scaling zero leaves it at zero.

(b) No—the sum of two of them may be off the axis (consider $\vec{e}_1 + \vec{e}_2$).

(c) Yes—in class, it was shown the vectors mapped to zero by one matrix form a subspace, and if a vector is in both *nullspaces*, its multiples and sums with other vectors with that property will be in each one separately, and so be in both of them.

(d) Yes—by trig identities, the phase change depends on the input phases, and the sum of rational numbers is rational.

(e) No—consider the phases π and $\pi + 1$.

(f) No—consider scaling by e.

Why do we want sets to be vector spaces? In some sense, vector spaces all work in the same way, so if we can show that some set we're interested in is a vector space, we get to import all kinds of results "for free." We're taking something we know how to work with—$\mathbb{R}^n$—and leveraging it to get answers to things that are harder to deal with—like differential equations (see future math courses).