Vector Spaces

The Punch Line: The same ideas we've been using translate to work on more abstract vector spaces, which describe many things which occur in "nature" (at least, in the mathematics we use to describe nature).

The Rules: A *vector space* is a set of objects V that satisfy these 10 axioms:

1. $\vec{x}+\vec{y} \in V$ (we say V is closed under addition)
2. $\vec{x}+\vec{y} = \vec{y}+\vec{x}$ (addition is commutative)
3. $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$ (addition is associative)
4. There is a $\vec{0}$ with the property that $\vec{x}+\vec{0} = \vec{x} = \vec{0}+\vec{x}$
5. There is a $\overrightarrow{-x}$ for every $\vec{x}$ so $\vec{x}+\overrightarrow{-x} = \vec{0}$
6. $c\vec{x} \in V$ (V is closed under scalar multiplication)
7. $c(\vec{x}+\vec{y}) = c\vec{x}+c\vec{y}$ (left distributivity)
8. $(c+d)\vec{x} = c\vec{x}+d\vec{x}$ (right distributivity)
9. $c(d\vec{x}) = (cd)\vec{x}$ (scaling is associative)
10. $1\vec{x} = \vec{x}$ (multiplicative identity)

1 Are these things vector spaces?

(a) The subset $\left\{\vec{0}\right\}$ in any $\mathbb{R}^n$

(b) $\mathbb{R}^2$ but scalar multiplication $c\vec{x}$ is defined as $c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x/c \\ y/c \end{bmatrix}$.

(c) $\mathbb{R}^2$ but addition is defined as $\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} y+w \\ x+z \end{bmatrix}$

(d) All functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$

(e) The set of all functions of the form $f(\theta) = a\sin(\theta+\phi)$

(f) The set of vectors $\vec{b}$ such that $\vec{b} = A\vec{x}$ (where A is fixed)

Subspaces: A subset U of a vector space V is a *subspace* if it contains $\vec{0}$ and is closed under addition and scaling. A subspace is a vector space in its own right.

2 Are these subsets subspaces?

(a) The vectors in $\mathbb{R}^3$ whose entries sum to zero

(b) The vectors in $\mathbb{R}^2$ which lie on one of the axes

(c) The vectors in $\mathbb{R}^3$ that are mapped to zero by matrices A and B

(d) The functions of the form $f(\theta) = A\sin(\theta + \phi)$ with ϕ rational

(e) The functions of the form $f(\theta) = A\sin(\theta + \phi)$ with ϕ irrational

(f) The functions of the form $f(\theta) = A\sin(\theta + \phi)$ with A rational

Why do we want sets to be vector spaces? In some sense, vector spaces all work in the same way, so if we can show that some set we're interested in is a vector space, we get to import all kinds of results "for free." We're taking something we know how to work with—$\mathbb{R}^n$—and leveraging it to get answers to things that are harder to deal with—like differential equations (see future math courses).