Math 4A
Key 4.2
Lecture 11/06/17

# Column and Null Spaces

**The Punch Line:** The sets of vectors we've been most interested in so far in the course—solution sets (to homogeneous systems) and spans—are in fact subspaces!

**Warm-Up:** Can these situations happen?

(a) A vector $\vec{x}$ is in both the null space and column space of a $3 \times 5$ matrix

(b) A vector $\vec{x}$ is in both the null space and column space of a $2 \times 2$ matrix

(c) A vector $\vec{x}$ is in neither the null space nor column space of a $2 \times 2$ matrix

(d) A vector $\vec{x}$ is in neither the null space nor column space of an invertible $4 \times 4$ matrix

(a) No, the null space is a subspace of $\mathbb{R}^5$ and the column space is a subspace of $\mathbb{R}^3$—they live in different "universes."

(b) Yes—consider the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(c) Yes—consider the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(d) No—for an invertible matrix, the columns span $\mathbb{R}^n$, so the column space includes every vector.

**Null Spaces:** The *null space* (also called the *kernel*) of a linear transformation $T$ in the vector space $V$ is the set of all vectors $\vec{x}$ that are mapped to $\vec{0} \in V$ by $T$: $T(\vec{x}) = \vec{0}$. For $\mathbb{R}^n$ and $T(\vec{x}) = A\vec{x}$ for a matrix $A$, we can explicitly describe the vectors in the null space by finding a parametric form for the solution set of the homogeneous equation $A\vec{x} = \vec{0}$. The vectors attached to each parameter span the null space.

**1** Describe the null spaces of the following linear transformations:

(a) $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\vec{x}$

(b) $T(\vec{x}) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 1 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix}\vec{x}$

(c) $T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\vec{x}$

(d) $T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\vec{x}$

(e) $T(\vec{x}) = \begin{bmatrix} 8 & 6 & 7 & 5 \\ 3 & 0 & 9 & 9 \end{bmatrix}\vec{x}$

(f) $T(\vec{x}) = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -1 & 2 \\ 0 & 0 \end{bmatrix}\vec{x}$

(g) $T(f(x)) = f(x) - f(0)$ acting on the space of all $\mathbb{R} \to \mathbb{R}$ functions*

*This is something of a challenge problem; it should help you understand null spaces, but it probably won't be on an exam.

(a) We find the REF of the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, from which we read out the solution set as $\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = s \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (I've chosen $s = -x_2$ to make the solution set look a little nicer). So, the null space is Span$\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$.

(b) Here the REF is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so the solution set is $\vec{x} = x_4 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$, so the null space is Span$\left\{\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}\right\}$.

(c) This matrix is invertible, so the solution set of the homogeneous equation is just the zero matrix, so the null space is the subspace consisting only of the zero vector.

(d) The matrix is already in REF, so we read off the solution set as $\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus, the null space is Span$\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.

(e) The REF here is $\begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & -\frac{17}{6} & -\frac{19}{6} \end{bmatrix}$. Then the solution set is $\vec{x} = x_3 \begin{bmatrix} -3 \\ \frac{17}{6} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ \frac{19}{6} \\ 0 \\ 1 \end{bmatrix} = s \begin{bmatrix} -18 \\ 17 \\ 6 \\ 0 \end{bmatrix} + t \begin{bmatrix} -18 \\ 19 \\ 0 \\ 6 \end{bmatrix}$. Thus, the null space is the two-dimensional space in $\mathbb{R}^4$ described as Span$\left\{\begin{bmatrix} -18 \\ 17 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} -18 \\ 19 \\ 0 \\ 6 \end{bmatrix}\right\}$.

(f) The REF here is $\begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, so the solution set is $\vec{x} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, so the null space is Span$\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$.

(g) Here we can't appeal to a REF, because we don't have a matrix. But, we *can* consider that if $T(f(x)) = 0$, then $f(x) - f(0) = 0$, or $f(x) = f(0)$. This means that the null space of $T$ here is the set of all functions which, for every $x$, give the same answer as at $x = 0$—that is, constant functions. Thus, the null space is the span of the constant function $f_1(x) = 1$.

**Column Spaces and Range:** The *column space* of a matrix is the span of its columns. For more general linear transformations, the analogous concept is *range*—the set of vectors in the vector space $V$ that can be reached by applying the linear transformation. In $\mathbb{R}^n$, we can get the column space as just the span of the columns (although we can describe it more succinctly if we eliminate linearly dependent columns).

> 2 Describe the range of these linear transformations. What is their dimension? Try to find a spanning set with only that many vectors. See if you can relate these situations to the null spaces you found on the last page.
>
> (a) $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\vec{x}$
>
> (b) $T(\vec{x}) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 1 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix}\vec{x}$
>
> (c) $T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\vec{x}$
>
> (d) $T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\vec{x}$
>
> (e) $T(\vec{x}) = \begin{bmatrix} 8 & 6 & 7 & 5 \\ 3 & 0 & 9 & 9 \end{bmatrix}\vec{x}$
>
> (f) $T(\vec{x}) = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -1 & 2 \\ 0 & 0 \end{bmatrix}\vec{x}$
>
> (g) $T(f(x)) = f(x) - f(0)$ acting on the space of all $\mathbb{R} \to \mathbb{R}$ functions*
>
> *This is again a challenge problem. What could the dimension be here?

(a) The range here is $\operatorname{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$—we don't need to write it twice. This is a distinct subspace of $\mathbb{R}^2$ from the null space, and we can see that any vector in $\mathbb{R}^2$ has a part in the column space and a part in the null space (that is, $\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ spans all of $\mathbb{R}^2$, so every vector is a linear combination of vectors from both subspaces). This is a nice property, and not one that always holds.

(b) Since columns one, two, and three are pivot columns, they are linearly independent (you can check this), so the column space is $\operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \\ 2 \end{bmatrix}\right\}$. It is three-dimensional (there are three pivot rows), and a little checking shows that the null space and the column space again don't intersect.

(c) The two columns are linearly independent, so their span is all of $\mathbb{R}^2$.

(d) The span of the columns is $\operatorname{Span}\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$, which is one-dimensional. Note that this is precisely the null space of this transformation! The subspace $\operatorname{Span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is *neither* in the null space nor column space of the matrix. This is a rather weird situation, but it shows that while the *sizes* of the null and column spaces match up to describe the size of the whole domain, they don't necessarily describe it themselves.

(e) The REF has two pivots, so the column space is all of $\mathbb{R}^2$. Just because there's a nontrivial null space doesn't mean there's anything missing from the range, if the domain is in a "bigger" (higher dimensional) vector space.

(f) Since the columns are linearly dependent, we only need one to describe the column space as $\operatorname{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}\right\}$, which is one-dimensional. This means that even though the null space is "small" compared to the target space, the range can be as well if the domain is lower-dimensional compared to the target space.

(g) We can see that any function with $f(0) = 0$ is unchanged by $T$, so it is in the range. If $f(0) \neq 0$, then $T(f(x))$ is zero at $x = 0$. This means that all and only functions which are zero at $x = 0$ are in the range of $T$.

What's going on with the linear transformation in part (d)? When (part of) the column space is in the null space, the matrix is sending vectors somewhere it will send to zero. If we applied the transformation twice (or, in general, enough times), it would send all vectors to zero. It's kind of a drawn-out process: send vectors matching some description (in some span) to zero, then change other vectors to take their places. It's important to remember that the null space is describing where vectors are *before* the transformation, while the column space is describing *after*.