Bases

The Punch Line: We have an efficient way to define subspaces using collections of vectors in them.

Warm-Up: Are these sets linearly independent? What do they span?

(a) $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\2\end{bmatrix},\begin{bmatrix}1\\2\\2\end{bmatrix}\right\}\subset\mathbb{R}^3$

(b) All vectors in $\mathbb{R}^{42}$ with a zero in at least one component

(c) $\left\{\begin{bmatrix}2\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\-1\\1\end{bmatrix},\begin{bmatrix}-2\\2\\-3\\2\end{bmatrix}\right\}\subset\mathbb{R}^4$

(d) $\left\{1, t-1, (t-1)^2+2(t-1)\right\}\subset\mathscr{P}_2$

(a) These are linearly independent. We can check this by examining the homogeneous linear equation $\begin{bmatrix}1&1&1\\1&1&2\\1&2&2\end{bmatrix}\vec{x}=\vec{0}$, or by observing that no linear combination of the first two can have different first and second component, no linear combination of the first and third can have different second and third components, and no linear combination of the last two can have all components the same. They span $\mathbb{R}^3$, as we can check by showing the inhomogeneous equation $\begin{bmatrix}1&1&1\\1&1&2\\1&2&2\end{bmatrix}\vec{x}=\vec{b}$ is consistent for all $\vec{b}$.

(b) This set is super dependent—it contains multiples of every vector in it. It spans $\mathbb{R}^4 2$, though, as it contains the vectors $\vec{e}_1,\vec{e}_2,\dots,\vec{e}_{42}$ which have a 1 in the position in their subscript and zeros elsewhere, and this subset spans $\mathbb{R}^4 2$.

(c) This set is linearly dependent—the last column is twice the second minus the first. It spans a plan in $\mathbb{R}^2$, in particular $s\begin{bmatrix}2\\0\\1\\0\end{bmatrix}+t\begin{bmatrix}0\\1\\-1\\1\end{bmatrix}$. This is because eliminating vectors linearly dependent on the rest doesn't change the span (because by assumption you can get at them with a linear combination of the rest), and the first two vectors are linearly independent.

(d) This set is linearly independent—the third entry is the only one with a t^2, so it is not dependent on either of the previous two, and they can't depend on it (there would be no way to eliminate that term). Clearly, the first two are linearly independent.

Bases: A *basis* for a vector space is a linearly independent spanning set. Every finite spanning set contains a basis by removing linearly dependent vectors, and many finite linearly independent sets may be extended to be a basis by adding vectors (if eventually this process terminates in a spanning set).

1 Are these sets bases for the indicated vector spaces? If not, can vectors be removed (which?) or added (how many?) to make it a basis?

(a) $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\2\end{bmatrix},\begin{bmatrix}1\\2\\2\end{bmatrix}\right\} \subset \mathbb{R}^3$

(b) $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}3\\2\end{bmatrix}\right\} \subset \mathbb{R}^2$

(c) $\left\{\begin{bmatrix}2\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\-1\\1\end{bmatrix},\begin{bmatrix}-2\\2\\-3\\2\end{bmatrix}\right\} \subset \mathbb{R}^4$

(d) $\left\{(t-1),(t-1)^2,(t-1)^3\right\} \subset \mathscr{P}_3$

(a) We saw previously that this set spans $\mathbb{R}^3$ and is linearly independent, so it is a basis.

(b) This set spans $\mathbb{R}^2$ (we could choose a coefficient on the second vector to match the second component of a given vector, then choose a coefficient on the first vector to match the first). It isn't linearly independent, though, as the second two vectors are linear combinations of the first. We could remove them to get a linearly independent set, hence a basis. In fact, no pair of vectors is linearly dependent, so any pair from this collection is a basis (we have to check that they span $\mathbb{R}^2$ on their own, but this is true in this case).

(c) We saw that this set is neither linearly independent nor spans $\mathbb{R}^4$. We could find a basis for $\mathbb{R}^4$ by removing the last vector (which is in the span of the first two) then adding in the vectors $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$. The resulting collection $\left\{\begin{bmatrix}2\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\-1\\1\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right\}$ is a basis, as you can check.

(d) This collection doesn't span $\mathscr{P}_3$ (you can verify that 1 is not in their span), but is linearly independent. We can simply add in 1, and verify that the result is indeed a basis by showing that if $p(x) = a + bx + cx^2 + dx^3$, then a linear combination of polynomials in this set yields p. Since this covers every polynomial in $\mathscr{P}_3$, we're good.

Finding Bases in $\mathbb{R}^n$: We're often interested in subspaces of the form Nul A and Col A for some matrix A. Fortunately, we can extract both by examining the Reduced Echelon Form of A.

A basis for Col A consists of all columns in A itself which correspond to pivot columns in the REF of A. A basis for Nul A consists of the vector parts corresponding to each free variable in a parametric vector representation of the solution set of the homogeneous equation $A\vec{x} = \vec{0}$, which we can find from the REF of A. Caution: In general, although free variables correspond to non-pivot columns in the REF, the basis for Nul A will *not* consist of those columns—in fact, they will often be of the wrong size!

2 Find bases for Nul A and Col A for each matrix below:

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a \neq 0$

(a) This is in REF, so we identify the pivot columns, and see they are a basis for Col A, $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$. The parametric vector form for the solution set is $s\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, so Nul A has basis $\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

(b) This has REF $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$, so a basis for Col A is $\left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}\right\}$, and a basis for Nul A is $\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}$. Note the different sizes of the vectors.

(c) The REF here is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. The column space basis is $\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}\right\}$, while the null space is $\{\vec{0}\}$, so there is no linearly independent spanning set.

(d) If $ad - bc \neq 0$, then the REF is I_2, so a basis for the column space is $\left\{\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right\}$, and the null space is $\{\vec{0}\}$. Otherwise, since $a \neq 0$ the REF is $\begin{bmatrix} 1 & b/a \\ 0 & 0 \end{bmatrix}$, so a basis for the column space is $\left\{\begin{bmatrix} a \\ c \end{bmatrix}\right\}$ and for the null space $\left\{\begin{bmatrix} -b \\ a \end{bmatrix}\right\}$.