Math 4A
Worksheet 4.3

Bases

The Punch Line: We have an efficient way to define subspaces using collections of vectors in them.

Warm-Up: Are these sets linearly independent? What do they span?

(a) $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}\right\} \subset \mathbb{R}^3$

(b) All vectors in $\mathbb{R}^{42}$ with a zero in at least one component

(c) $\left\{\begin{bmatrix}2\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}-2\\2\\-3\\2\end{bmatrix}\right\} \subset \mathbb{R}^4$

(d) $\left\{1, t-1, (t-1)^2 + 2(t-1)\right\} \subset \mathscr{P}_2$

Bases: A *basis* for a vector space is a linearly independent spanning set. Every finite spanning set contains a basis by removing linearly dependent vectors, and many finite linearly independent sets may be extended to be a basis by adding vectors (if eventually this process terminates in a spanning set).

1 Are these sets bases for the indicated vector spaces? If not, can vectors be removed (which?) or added (how many?) to make it a basis?

(a) $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\2\end{bmatrix},\begin{bmatrix}1\\2\\2\end{bmatrix}\right\}\subset\mathbb{R}^3$

(b) $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}3\\2\end{bmatrix}\right\}\subset\mathbb{R}^2$

(c) $\left\{\begin{bmatrix}2\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\-1\\1\end{bmatrix},\begin{bmatrix}-2\\2\\-3\\2\end{bmatrix}\right\}\subset\mathbb{R}^4$

(d) $\left\{(t-1),(t-1)^2,(t-1)^3\right\}\subset\mathscr{P}_3$

Finding Bases in $\mathbb{R}^n$: We're often interested in subspaces of the form Nul A and Col A for some matrix A. Fortunately, we can extract both by examining the Reduced Echelon Form of A.

A basis for Col A consists of all columns in A itself which correspond to pivot columns in the REF of A. A basis for Nul A consists of the vector parts corresponding to each free variable in a parametric vector representation of the solution set of the homogeneous equation $A\vec{x} = \vec{0}$, which we can find from the REF of A. Caution: In general, although free variables correspond to non-pivot columns in the REF, the basis for Nul A will *not* consist of those columns—in fact, they will often be of the wrong size!

2 Find bases for Nul A and Col A for each matrix below:

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a \neq 0$