Coordinates

The Punch Line: If we have a basis of n vectors for any vector space, we can describe (and work with) any vector from the space or equation in it as if it were in $\mathbb{R}^n$ all along!

Coordinate Vectors: If we have an *ordered* basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ for vector space V, then any vector $v \in V$ has a unique representation

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n,$$

where each c_i is a real number. Then we can write the *coordinate vector* $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$.

1 Find the representation of the given vector $\vec{v}$ with respect to the ordered basis $\mathcal{B}$.

(a) $\mathcal{B} = \left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$, $\vec{v} = \begin{bmatrix}8\\0\\5\end{bmatrix}$

(b) $\mathcal{B} = \left\{1, t, t^2, t^3\right\}$, $\vec{v} = t^3 - 2t^2 + t$

(c) $\mathcal{B} = \left\{1, (t-1), (t-1)^2, (t-1)^3\right\}$, $\vec{v} = t^3 - 2t^2 + t$

(d) $\mathcal{B} = \left\{\begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$, $\vec{v} = \begin{bmatrix}1\\1\\-2\end{bmatrix}$

(e) $\mathcal{B} = \left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\right\}$,
$\vec{v} = \begin{bmatrix}2 & 1\\-1 & 0\end{bmatrix}$

(a) Here we have the second original component first, followed by the third original component, followed by the first original. Thus, $\begin{bmatrix}8\\0\\5\end{bmatrix} = \begin{bmatrix}0\\5\\8\end{bmatrix}_{\mathcal{B}}$.

(b) Here, we get the coordinate vector $\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}_{\mathcal{B}}$.

(c) We can rearrange our polynomial as $t^3 - 2t^2 + t = (t-1)^2 + (t-1)^3$, so its coordinates in this basis are $\begin{bmatrix}1\\1\\0\\0\end{bmatrix}_{\mathcal{B}}$.

(d) We can see that to match the middle component, we need $c_1 = -\frac{1}{2}$. This leaves $\begin{bmatrix}3/2\\0\\-3/2\end{bmatrix}$, so $c_2 = \frac{3}{2}$ and $\begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}-1/2\\3/2\end{bmatrix}_{\mathcal{B}}$. This raises the important point that *the number of entries in a coordinate vector depends on the length of the basis it relates to*, not *the original vector space*!

Change of Coordinates in $\mathbb{R}^n$: If we have a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$, we can recover the standard representation by using the matrix P whose columns are the (ordered) basis elements represented in the standard basis:

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}.$$

The matrix P^{-1} takes vectors in the standard encoding and represents them with respect to $\mathcal{B}$. Thus, if $\mathcal{C}$ is another basis for the same space and Q is the matrix bringing representations with respect to $\mathcal{C}$ to the standard basis, then $Q^{-1}P$ is a matrix which takes a vector encoded with respect to $\mathcal{B}$ and returns its encoding with respect to $\mathcal{C}$. That is,

$$[\vec{v}]_{\mathcal{C}} = Q^{-1}P\,[\vec{v}]_{\mathcal{B}}.$$

2 Compute the change of basis matrices for the following bases (into and from the standard basis).

(a) $\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ (c) $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$ (d) $\left\{\begin{bmatrix}2\\5\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix}\right\}$

(a) We have $P = \begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix}$, and $P^{-1} = P$ (which we can see as P just transposes the first and third components).

(b) We have $P = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$, and $P^{-1} = \begin{bmatrix}1 & -1 & -1\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix}$ (check this!).

(c) We have $P = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$ and $P^{-1} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$.

(d) We have $P = \begin{bmatrix}2 & 1\\5 & 3\end{bmatrix}$ and $P^{-1} = \begin{bmatrix}3 & -1\\-5 & 2\end{bmatrix}$.

3 Compute the change of basis matrices between the two bases:

(a) $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(b) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}, \mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$

(a) The transition from encoding in $\mathcal{B}$ to $\mathcal{C}$ is given by $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Its inverse is $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

(b) The transition from $\mathcal{B}$ to $\mathcal{C}$ is given by $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -3 & -7 \end{bmatrix}$. Its inverse is $\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 7 & 4 \\ -3 & -2 \end{bmatrix}$.