Dimension and Rank

The Punch Line: We can compare the "size" of different vector spaces and subspaces by looking at the size of their bases.

Warm-Up: Are these bases for the given vector space?

(a) $\left\{\begin{bmatrix}2\\5\end{bmatrix}, \begin{bmatrix}5\\-2\end{bmatrix}\right\}$ in $\mathbb{R}^2$

(b) $\{1, 1+t, t\}$ in $\mathscr{P}_1$

(c) $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right\}$ in the vector space (you can check it is one) of all 2×2 matrices

(d) $\left\{1, t, t^2, \dots, t^n\right\}$ for some fixed n in the space of all polynomials

(a) Yes, they are linearly independent, and span $\mathbb{R}^2$.

(b) No, they are not linearly independent—$1 + t$ is the sum of two other basis elements.

(c) No, they do not span the space of all matrices—any matrix with a nonzero bottom right entry isn't in their span.

(d) No, because t^{n+1} is not in their span.

Dimension: If one basis for a vector space V has n vectors, then all others do. We can see this by writing the other basis' coordinates with respect to the first basis, then looking at the Reduced Echelon Form of this matrix—there can't be any free variables, and there must be n pivots, so there must be n vectors in the new basis.

1 Find the dimension for each of the following subspaces.

(a) $\text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

(b) $\text{Span}\left\{1-t+t^2, 1+t-2t^2, t^2-t, t^3-t, t^3-t^2\right\}$

(c) $\text{Nul}\begin{bmatrix}1 & 1 & 1\\1 & -1 & 1\\-1 & 1 & -1\end{bmatrix}$

(d) $\text{Col}\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\-1 & 0 & 1\end{bmatrix}$

(a) These are linearly independent, and clearly span their span, so the span has a basis of size 2, so is of dimension 2.

(b) We look at the coordinates of this with respect to the basis $\left\{1, t, t^2, t^3\right\}$. This gives the vectors $\left\{\begin{bmatrix}1\\-1\\1\\0\end{bmatrix},\begin{bmatrix}1\\1\\-2\\0\end{bmatrix},\begin{bmatrix}0\\-1\\1\\0\end{bmatrix},\begin{bmatrix}0\\-1\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\-1\\1\end{bmatrix}\right\}$.

We examine the matrix $\begin{bmatrix}1 & 1 & 0 & 0 & 0\\-1 & 1 & -1 & -1 & 0\\1 & -2 & 1 & 0 & -1\\0 & 0 & 0 & 1 & 1\end{bmatrix}$, and find that it has REF $\begin{bmatrix}1 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & -1\\0 & 0 & 0 & 1 & 1\end{bmatrix}$. Thus, the first four vectors are linearly independent and span the whole span, so they are a basis. Thus, this span has dimension 4 (and is therefore all of $\mathscr{P}_3$).

(c) The REF here is $\begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$, so there is one free variable, so the null space has dimension 1.

(d) The REF here is $\begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}$, which has two pivots, so the column space has dimension 2.

Rank of a Matrix: For any $n \times m$ matrix A, the dimension of the null space is the number of free variables and the dimension of the column space is the number of pivots. These add up to m, the number of columns (a column is either a pivot or corresponds to a free variable). We call the dimension of the column space the *rank* of a matrix.

2 Find the ranks of the following matrices:

(a) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 6 \end{bmatrix}$

(d) An invertible $n \times n$ matrix

(a) This matrix clearly has two pivots, so the column space will have dimension 2, so this is the rank of the matrix.

(b) We can see that these columns are all linearly independent, so form a basis for their span, so the rank is 3.

(c) We can see through interchanging rows that this matrix has 3 pivots, hence has rank 3.

(d) One equivalent property to being invertible is having no free variables, which means that $\dim \mathrm{Nul} A = 0$, so the dimension of the column space must be the number of columns n, so this must be the rank.

The statement that $\operatorname{rank} A + \dim \operatorname{Nul} A$ is the number of columns of A is an important theorem known as the Rank Nullity Theorem (some people call $\dim \operatorname{Nul} A$ the *nullity* of A). It is basically saying that the input space to A has only two important parts: the null space, and the vectors which contain the information for knowing what the column space looks like. There's a bit more to it than that, but the gist is there isn't some third kind of vector lurking around that isn't related to either the null or column spaces.