Dimension and Rank

The Punch Line: We can compare the "size" of different vector spaces and subspaces by looking at the size of their bases.

Warm-Up: Are these bases for the given vector space?

(a) $\left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \end{bmatrix} \right\}$ in $\mathbb{R}^2$

(b) $\{1, 1+t, t\}$ in $\mathscr{P}_1$

(c) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ in the vector space (you can check it is one) of all 2×2 matrices

(d) $\left\{1, t, t^2, \dots, t^n\right\}$ for some fixed n in the space of all polynomials

Dimension: If one basis for a vector space V has n vectors, then all others do. We can see this by writing the other basis' coordinates with respect to the first basis, then looking at the Reduced Echelon Form of this matrix—there can't be any free variables, and there must be n pivots, so there must be n vectors in the new basis.

1 Find the dimension for each of the following subspaces.

(a) $\text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

(b) $\text{Span}\left\{1-t+t^2, 1+t-2t^2, t^2-t, t^3-t, t^3-t^2\right\}$

(c) $\text{Nul}\begin{bmatrix}1 & 1 & 1\\1 & -1 & 1\\-1 & 1 & -1\end{bmatrix}$

(d) $\text{Col}\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\-1 & 0 & 1\end{bmatrix}$

Rank of a Matrix: For any $n \times m$ matrix A, the dimension of the null space is the number of free variables and the dimension of the column space is the number of pivots. These add up to m, the number of columns (a column is either a pivot or corresponds to a free variable). We call the dimension of the column space the *rank* of a matrix.

2 Find the ranks of the following matrices:

(a) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 6 \end{bmatrix}$

(d) An invertible $n \times n$ matrix

The statement that $\operatorname{rank} A + \dim \operatorname{Nul} A$ is the number of columns of A is an important theorem known as the Rank Nullity Theorem (some people call $\dim \operatorname{Nul} A$ the *nullity* of A). It is basically saying that the input space to A has only two important parts: the null space, and the vectors which contain the information for knowing what the column space looks like. There's a bit more to it than that, but the gist is there isn't some third kind of vector lurking around that isn't related to either the null or column spaces.