Math 4A
Worksheet 5.2

Eigenvalues and Where to Find Them

The Punch Line: Finding the eigenvalues of a matrix boils down to finding the roots of a polynomial.

Warm-Up: What are the eigenvalues of these matrices? What is the dimension of each eigenspace?

[Note: you shouldn't have to do many computations here—just look at Echelon Forms and try to see how many free variables there will be.]

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 2 & 0 & 8 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

(e) $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

The Characteristic Equation: If λ is an eigenvalue of the matrix A, that means there is some nonzero $\vec{v} \in \mathbb{R}^n$ that satisfies the equation $A\vec{v} = \lambda\vec{v}$. Then $(A - \lambda I_n)\vec{v} = \vec{0}$ (from putting all terms with $\vec{v}$ on the same side), so $(A - \lambda I_n)$ is a non-invertible matrix (it has nontrivial null space, because $\vec{v} \neq \vec{0}$). Since we know that a matrix being not invertible is equivalent to its determinant being zero, we can check when the equation $\det(A - \lambda I_n) = 0$ is true. This gives a polynomial equation in λ of degree n (why?), so if we can find the roots of the polynomial, we know all of the eigenvalues. This equation is known as the *characteristic equation*.

1 What is the characteristic equation for each of these matrices?
[Note: You need not solve the characteristic equation yet.]

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} -3/2 & 1/2 \\ 1/2 & -3/2 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$

2 What are the eigenvalues of these matrices? What are the dimensions of each eigenspace?

[Note: Again, try to minimize computation—we're not after the eigenspace itself, just its dimension, so you only need to manipulate the matrix into an Echelon Form matrix, not fully solve for its null space.]

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} -3/2 & 1/2 \\ 1/2 & -3/2 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$

Under the Hood: You may have noticed we often got one-dimensional eigenspaces. If we know something is an eigenvalue, we know that its eigenspace is *at least* one dimensional, and the eigenspaces for different eigenvalues are distinct except for the zero vector (otherwise A would act on a vector in both by scaling by the different eigenvalues, which would give two different answers!). Thus, if we have n distinct eigenvalues for a matrix in $\mathbb{R}^n$, we know we have found n distinct subspaces, each of which is at least one-dimensional. This means they *have* to be one-dimensional, otherwise $\mathbb{R}^n$ would have more than n dimensions! As it turns out, the characteristic equation gives us information on the maximum size of each eigenspace, through the *multiplicity* of each root (how many times it appears).