Math 4A
Key 6.1

Inner Products, Length, and Orthogonality

The Punch Line: We can compute a real number relating two vectors—or a vector to itself—that gives information on both length and angle.

Warm-Up What are the lengths of these vectors, as found geometrically (using things like the Pythagorean Theorem)?

(a) $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(e) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(f) $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

(a) This vector is along one of the axes, so we can see that it's length is 3.

(b) Similarly for this one; although it is in the negative y direction, its length is positive 2.

(c) We can think of the vector as the sum of its x and y components, and because these are perpendicular, we can use the Pythagorean Theorem to see that the square of the length of the vector is the sum of the squares of the lengths of the legs. For this vector, it means the square of the length is the sum of $1^2 + 1^2 = 2$, so the length itself is $\sqrt{2}$. This should be familiar as the length of the diagonal of a square of side length 1.

(d) Here we get a right triangle formed by the vector and its x and y components. It's a Pythagorean Triple, so we see that the length squared is $3^2 + 4^2 = 5^2$, or the length is 5.

(e) Here we note that one leg is of length 1 and one of length 2, so the total length is $\sqrt{1^2 + 2^2} = \sqrt{5}$. Note again that we are only using positive lengths.

(f) Here we can work in two stages: the "footprint" in the xy plane is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, with length $\sqrt{2}$ as before. This then forms one leg of the vector, with the other leg being the z component. Then the length is $\sqrt{\left(\sqrt{2}\right)^2 + 3^2} = \sqrt{11}$. This is a rather cumbersome way to get the length, so we'd like to find a way to do it all in one step.

The Inner Product: If we think about a vector $\vec{v} \in \mathbb{R}^n$ as a $n \times 1$ matrix (a single column), then $\vec{v}^T$ is a $1 \times n$ matrix (a single row, sometimes called a row vector). Then we can multiply $\vec{v}^T$ against a vector (on the left) to get a 1×1 matrix, which we can consider a scalar. This is the idea behind the *inner product* in $\mathbb{R}^n$, also called the *dot product*: we take two vectors, $\vec{u}$ and $\vec{v}$, and define their inner product as $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$. This corresponds to multiplying together corresponding entries in the vectors, then adding all of the results to get a single number.

1 Find the inner product of the two given vectors:

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(e) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$

(f) $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} -y \\ x \end{bmatrix}$

(a) Here we get $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = (1)(2) + (1)(3) = 5.$

(b) Here we get $\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 + 0 = 1.$

(c) Here $(1)(3) + (-1)(2) + (1)(-1) + (-2)(0) = 3 - 2 - 1 + 0 = 0.$

(d) Here $(1)(0) + (0)(1) = 0.$

(e) Here $(0)(x) + (0)(y) = 0.$

(f) Here $(x)(y) + (-y)(x) = xy - yx = 0.$

Length and Orthogonality: We observe that in $\mathbb{R}^2$, the quantity $\sqrt{\vec{v}\cdot\vec{v}}$ is the length of $\vec{v}$ as given by the Pythagorean Theorem. This motivates us to define the length of a vector in *any* $\mathbb{R}^n$ as $\left\|\vec{v}\right\| = \sqrt{\vec{v}\cdot\vec{v}}$ (encouraged that it also agrees with our idea of length in $\mathbb{R}^1$ and $\mathbb{R}^3$). Then the *distance* between $\vec{u}$ and $\vec{v}$ is $\left\|\vec{u}-\vec{v}\right\|$, the length of the vector between them.

We also observe that in $\mathbb{R}^2$, if $\vec{u}$ and $\vec{v}$ are perpendicular then $\vec{u}\cdot\vec{v} = 0$, and vice versa. To generalize this, we say $\vec{u}$ and $\vec{v}$ are *orthogonal* if $\vec{u}\cdot\vec{v} = 0$ (and indeed, this matches with perpendicularity in three dimensions as well).

2 What are the lengths of these vectors (computed with inner products)?

(a) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(b) $\begin{bmatrix} 2 \\ -3 \\ 1 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(d) The vector of all 1s in $\mathbb{R}^n$ (this is something of a challenge problem)

(a) Here we get $\begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = (3)(3)+(4)(4) = 25$ as the inner product, so the length is $\sqrt{25} = 5$.

(b) Here the inner product is $(2)^2+(-3)^2+(1)^2+(-1)^2 = 4+9+1+1 = 15$, so the length is $\sqrt{15} \approx 3.87$ (the decimal expansion isn't necessary to compute).

(c) Here the inner product is $(1)^2+(1)^2+(1)^2 = 3(1)^2 = 3$, so the length is $\sqrt{3} \approx 1.73$.

(d) Here the inner product will be n copies of $(1)^2$ summed, which will come out to be n. Thus, the vector will have length $\sqrt{n}$.

3 What is the distance between these two vectors? Are they orthogonal?

(a) $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$

(b) $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}1\\2\\-3\end{bmatrix}$

(c) $\begin{bmatrix}2\\5\end{bmatrix}$ and $\begin{bmatrix}-2\\-5\end{bmatrix}$

(d) Two (different) standard basis vectors in $\mathbb{R}^n$

(a) The distance between these is the length of $\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$. This is $\sqrt{(0)^2+(2)^2} = 2$. Their dot product is $\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 0$, so they are orthogonal.

(b) The distance here is the length of their difference $\begin{bmatrix}0\\0\\6\end{bmatrix}$, which is 6. The dot product is $\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}1\\2\\-3\end{bmatrix} = 1+4-9 = -5 \neq 0$, so they are not orthogonal.

(c) The distance here is the length of $\begin{bmatrix}4\\10\end{bmatrix}$, which is $\sqrt{16+100} = \sqrt{116} = 2\sqrt{29}$. Note that the length of each vector is $\sqrt{29}$. Their inner product is $\begin{bmatrix}2 & 5\end{bmatrix}\begin{bmatrix}-2 & -5\end{bmatrix} = -4-25 = -29 \neq 0$, so they are not orthogonal.

(d) The distance here is the length of a vector which has two nonzero entries: one 1 and one -1. Thus, it is $\sqrt{1^2+(-1)^2} = \sqrt{2}$. Since the 1 in one standard basis vector will multiply against a 0 in the other, the inner product will be zero, and thus the standard basis vectors are orthogonal.

Under the Hood: This idea of orthogonality can be used to find the collection of *all* vectors which are orthogonal to some given $\vec{u}$. These are the solutions to the equation $\vec{u}\cdot\vec{v} = \vec{u}^T\vec{v} = 0$. This is just finding the nullspace of the matrix $\vec{u}^T$, but now it has a nice geometric interpretation. The solution set is a subspace, known as the *orthogonal complement* of $\vec{u}$.