Math 4A
Worksheet 6.1

Inner Products, Length, and Orthogonality

The Punch Line: We can compute a real number relating two vectors—or a vector to itself—that gives information on both length and angle.

Warm-Up What are the lengths of these vectors, as found geometrically (using things like the Pythagorean Theorem)?

(a) $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(e) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(f) $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

The Inner Product: If we think about a vector $\vec{v} \in \mathbb{R}^n$ as a $n \times 1$ matrix (a single column), then $\vec{v}^T$ is a $1 \times n$ matrix (a single row, sometimes called a row vector). Then we can multiply $\vec{v}^T$ against a vector (on the left) to get a 1×1 matrix, which we can consider a scalar. This is the idea behind the *inner product* in $\mathbb{R}^n$, also called the *dot product*: we take two vectors, $\vec{u}$ and $\vec{v}$, and define their inner product as $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$. This corresponds to multiplying together corresponding entries in the vectors, then adding all of the results to get a single number.

1 Find the inner product of the two given vectors:

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(e) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$

(f) $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} -y \\ x \end{bmatrix}$

Length and Orthogonality: We observe that in $\mathbb{R}^2$, the quantity $\sqrt{\vec{v}\cdot\vec{v}}$ is the length of $\vec{v}$ as given by the Pythagorean Theorem. This motivates us to define the length of a vector in *any* $\mathbb{R}^n$ as $\left\|\vec{v}\right\| = \sqrt{\vec{v}\cdot\vec{v}}$ (encouraged that it also agrees with our idea of length in $\mathbb{R}^1$ and $\mathbb{R}^3$). Then the *distance* between $\vec{u}$ and $\vec{v}$ is $\left\|\vec{u}-\vec{v}\right\|$, the length of the vector between them.

We also observe that in $\mathbb{R}^2$, if $\vec{u}$ and $\vec{v}$ are perpendicular then $\vec{u}\cdot\vec{v} = 0$, and vice versa. To generalize this, we say $\vec{u}$ and $\vec{v}$ are *orthogonal* if $\vec{u}\cdot\vec{v} = 0$ (and indeed, this matches with perpendicularity in three dimensions as well).

2 What are the lengths of these vectors (computed with inner products)?

(a) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(b) $\begin{bmatrix} 2 \\ -3 \\ 1 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(d) The vector of all 1s in $\mathbb{R}^n$ (this is something of a challenge problem)

3 What is the distance between these two vectors? Are they orthogonal?

(a) $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$

(b) $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}1\\2\\-3\end{bmatrix}$

(c) $\begin{bmatrix}2\\5\end{bmatrix}$ and $\begin{bmatrix}-2\\-5\end{bmatrix}$

(d) Two (different) standard basis vectors in $\mathbb{R}^n$

Under the Hood: This idea of orthogonality can be used to find the collection of *all* vectors which are orthogonal to some given $\vec{u}$. These are the solutions to the equation $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = 0$. This is just finding the nullspace of the matrix $\vec{u}^T$, but now it has a nice geometric interpretation. The solution set is a subspace, known as the *orthogonal complement* of $\vec{u}$.