## Math 4A Worksheet 6.1 Inner Products, Length, and Orthogonality

**The Punch Line:** We can compute a real number relating two vectors—or a vector to itself—that gives information on both length and angle.

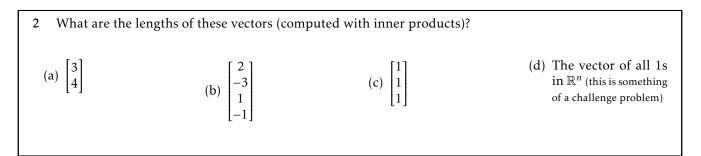
Warm-Up Theorem)?	What are the lengths of these vectors, as found	geometrically (using things like the Pythagorean
(a) $\begin{bmatrix} 3\\0 \end{bmatrix}$	(c) $\begin{bmatrix} 1\\1 \end{bmatrix}$	(e) $\begin{bmatrix} -1\\ 2 \end{bmatrix}$
(b) $\begin{bmatrix} 0\\ -2 \end{bmatrix}$	(d) $\begin{bmatrix} 3\\4 \end{bmatrix}$	(f) $\begin{bmatrix} 1\\1\\3 \end{bmatrix}$

**The Inner Product:** If we think about a vector  $\vec{v} \in \mathbb{R}^n$  as a  $n \times 1$  matrix (a single column), then  $\vec{v}^T$  is a  $1 \times n$  matrix (a single row, sometimes called a row vector). Then we can multiply  $\vec{v}^T$  against a vector (on the left) to get a  $1 \times 1$  matrix, which we can consider a scalar. This is the idea behind the *inner product* in  $\mathbb{R}^n$ , also called the *dot product*: we take two vectors,  $\vec{u}$  and  $\vec{v}$ , and define their inner product as  $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$ . This corresponds to multiplying together corresponding entries in the vectors, then adding all of the results to get a single number.

1 Find the inner product of the two given vectors:				
(a) $\begin{bmatrix} 1\\1 \end{bmatrix}$ and $\begin{bmatrix} 2\\3 \end{bmatrix}$	(c) $\begin{bmatrix} 1\\-1\\1\\-2 \end{bmatrix}$ and $\begin{bmatrix} 3\\2\\-1\\0 \end{bmatrix}$	(e) $\begin{bmatrix} 0\\0 \end{bmatrix}$ and $\begin{bmatrix} x\\y \end{bmatrix}$		
(b) $\begin{bmatrix} 1\\0 \end{bmatrix}$ and $\begin{bmatrix} 1\\1 \end{bmatrix}$	(d) $\begin{bmatrix} 1\\0 \end{bmatrix}$ and $\begin{bmatrix} 0\\1 \end{bmatrix}$	(f) $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} -y \\ x \end{bmatrix}$		

**Length and Orthogonality:** We observe that in  $\mathbb{R}^2$ , the quantity  $\sqrt{\vec{v} \cdot \vec{v}}$  is the length of  $\vec{v}$  as given by the Pythagorean Theorem. This motivates us to define the length of a vector in *any*  $\mathbb{R}^n$  as  $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$  (encouraged that it also agrees with our idea of length in  $\mathbb{R}^1$  and  $\mathbb{R}^3$ ). Then the *distance* between  $\vec{u}$  and  $\vec{v}$  is  $\|\vec{u} - \vec{v}\|$ , the length of the vector between them.

We also observe that in  $\mathbb{R}^2$ , if  $\vec{u}$  and  $\vec{v}$  are perpendicular then  $\vec{u} \cdot \vec{v} = 0$ , and vice versa. To generalize this, we say  $\vec{u}$  and  $\vec{v}$  are *orthogonal* if  $\vec{u} \cdot \vec{v} = 0$  (and indeed, this matches with perpendicularity in three dimensions as well).



3 What is the distance between these two vectors? Are they orthogonal? (a)  $\begin{bmatrix} 1\\1 \end{bmatrix}$  and  $\begin{bmatrix} 1\\-1 \end{bmatrix}$  (b)  $\begin{bmatrix} 1\\2\\3 \end{bmatrix}$  and  $\begin{bmatrix} 1\\2\\-3 \end{bmatrix}$  (c)  $\begin{bmatrix} 2\\5 \end{bmatrix}$  and  $\begin{bmatrix} -2\\-5 \end{bmatrix}$  (d) Two (different) standard basis vectors in  $\mathbb{R}^n$ 

**Under the Hood:** This idea of orthogonality can be used to find the collection of *all* vectors which are orthogonal to some given  $\vec{u}$ . These are the solutions to the equation  $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = 0$ . This is just finding the nullspace of the matrix  $\vec{u}^T$ , but now it has a nice geometric interpretation. The solution set is a subspace, known as the *orthogonal complement* of  $\vec{u}$ .