Systems of Linear Equations

The Punch Line: We can solve systems of linear equations by manipulating a matrix that represents the system.

Warm-Up: Which of these equations are linear?

(a) $y = mx + b$, with x and y as variables

(b) $(y - y_0) + 4(x - x_0) = 0$, with x and y as variables

(c) $4x + 2y - 9z = 12$, with x, y, and z as variables

(d) $x_1^2 + x_2^2 = 1$, with x_1 and x_2 as variables

(e) $a^2x + 3b^3y = 6$, with x and y as variables

(f) $a^2x + 3b^3y = 6$, with a and b as variables

The Setup: When we have a linear system of equations, we can make an *augmented matrix* representing the system by arranging the coefficients on the left side of the matrix (keeping them in the same order for each equation, and writing a 0 whenever a variable is missing from one of the equations), and the constants on the right side. This is often easiest to do when each equation is written in the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, so it can help to rewrite equations like this if they are given to you differently.

1. Write down an augmented matrix representing these linear systems.

(a) The system for x_1 and x_2 given by

$$\begin{aligned} x_1 + x_2 &= 4 \\ x_1 - 2x_2 &= 1 \end{aligned}$$

(b) The system for x, y, and z given by

$$\begin{aligned} x - 2y + z &= 0 \\ x + y &= 2 \\ y - z &= 1 \end{aligned}$$

(c) The system for x and y given by

$$\begin{aligned} y &= 4 - x \\ x + 1 &= 2y + 2 \end{aligned}$$

2. Write down a linear system of equations represented by these augmented matrices.

(a) $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ -3 & 1 & 0 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -2 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$

(c) $\left[\begin{array}{ccc|c} 1 & 2 & -4 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The Execution: Once we have a matrix representing a linear system of equations, we can use *elementary row operations* on the matrix to find equivalent system of equations. These operations are

1) Replacement: Replace a row with itself plus a multiple of a different row,

2) Interchange: Switch the order of two rows,

3) Scaling: Multiply everything in the row by the same constant (other than 0).

The goal is to use these three operations to find an equivalent system of equations that is easier to solve.

3. Solve each of these linear systems of equations.

(a) The two variable system given by

$$\begin{aligned} x_1 + x_2 &= 4 \\ x_1 - 2x_2 &= 1 \end{aligned}$$

(b) The three variable system given by

$$\begin{aligned} x - 2y + z &= 0 \\ x + y &= 2 \\ y - z &= 1 \end{aligned}$$

(c) The three variable system given by

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 - x_3 &= 0 \end{aligned}$$

Under the Hood: Why do elementary row operations result in equivalent systems of equations? Each equation is just some true statement about the solutions, and the system of equations is a collection of true statements that together give us enough information to figure out exactly what the solutions are. Elementary row operations are tools we use to make new true statements that contain the same amount of information about the solutions as the old ones. We know the new statements contain the same amount of information because they're reversible—if we started with them, we could do a different series of operations to get the original system. Come see me if you want to talk about why we are sure they make true statements—or try to prove it on your own!

The Row Reduction Algorithm

The Punch Line: Given any linear system of equations, there is a procedure which finds a particularly simple equivalent system.

Warm-Up: For each of these matrices, determine if it is in Echelon Form, Reduced Echelon Form, or neither.

(a)
$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

(b)
$$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(c)
$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(d)
$$\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(e)
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(f)
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Using the Algorithm: Five steps transform any matrix into a row-equivalent Reduced Echelon Form matrix:

1) Identify the pivot column. This will be the leftmost column with a nonzero entry.
2) Select a nonzero entry in that column to be the pivot for that column. If necessary, interchange rows to put it at the top of the matrix.
3) Eliminate all of the nonzero entries in the pivot column by using row replacement operations.
4) Repeat steps 1)-3) on all rows you haven't yet used.
5) Eliminate all nonzero entries above each pivot, and scale each nonzero row so its pivot is 1.

1. Transform each of these matrices into Reduced Echelon Form with the above procedure:

(a)

$$\begin{bmatrix} 1 & 1 & 1 \\ -2 & 4 & -2 \\ -1 & 0 & 1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 2 & 0 & 4 & 0 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 3 & 3 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

Interpreting the Results: The Reduced Echelon Form of the augmented matrix of a linear system can be used to find all solutions (the *solution set*) of the system at once. To do this, we write out the system corresponding to the Reduced Echelon Form matrix, then solve for all of the variables in pivot positions (we can do this easily because each one only appears in a single equation). Any remaining variables are called *free variables*, and can take on any value in a solution.

2. Find the solution set of each of these linear systems:

(a)

$$\begin{aligned} 2x_1 + 4x_3 &= 0 \\ x_2 + x_3 &= 3 \\ x_1 + x_2 + 3x_3 &= 3 \end{aligned}$$

(b)

$$\begin{aligned} x + y &= 1 \\ 4y - 2x &= -2 \\ -x &= 1 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 + 2x_2 &= 1 \\ 2x_1 + 3x_2 &= 3 \end{aligned}$$

Under the Hood: The Reduced Echelon Forms of any two equivalent systems are the same. Since equivalent systems have the same solution set (by definition!), it is in some sense the simplest system with that solution set. Thus, the Row Reduction Algorithm is a way to find the simplest description of the solution set of a linear system—that works every time!

Vector Equations

The Punch Line: Vector equations allow us to think about systems of linear equations as geometric objects, and are an efficient notation to work with.

Warm-Up: Sketch the following vectors in $\mathbb{R}^2$:

(a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

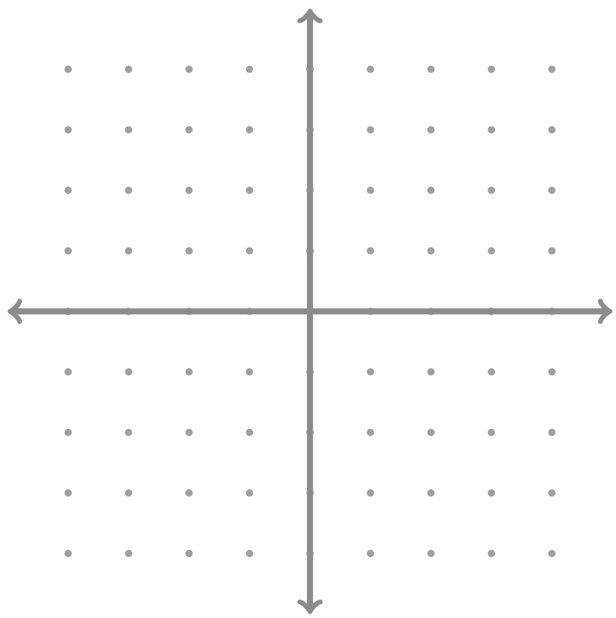
(b) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(f) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$



Linear Combinations: A *linear combination* of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ with *weights* $w_1, w_2, \ldots, w_n$ is the vector $\vec{y}$ defined by

$$\vec{y} = w_1\vec{v}_1 + w_2\vec{v}_2 + \cdots + w_n\vec{v}_n.$$

That is, it's a sum of multiples of the vectors. Geometrically, it corresponds to stretching each vector $\vec{v}_i$ (where i is one of $1, 2, \ldots, n$) by the weight w_i, then laying them end to end and drawing $\vec{y}$ to the endpoint of the last vector.

1 Compute the following linear combinations:

(a) $\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}$

(b) $(-1)\begin{bmatrix}1\\1\end{bmatrix}$

(c) $\begin{bmatrix}2\\3\end{bmatrix} - \begin{bmatrix}3\\2\end{bmatrix}$

(d) $\begin{bmatrix}1\\2\\3\end{bmatrix} - 2\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}1\\0\\-1\end{bmatrix}$

(e) $\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix}$

(f) $4\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix} - 2\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix} + 3\begin{bmatrix}\frac{2}{9}\\2\end{bmatrix}$

Think about what each of these linear combinations mean geometrically (try sketching them).

Span: The *span* of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is the set of all linear combinations of them. If $\vec{x}$ is in Span$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, then we will be able to find some weights $w_1, w_2, \ldots, w_n$ to make the linear combination using those weights result in $\vec{x}$:

$$w_1\vec{v}_1 + w_2\vec{v}_2 + \cdots + w_n\vec{v}_n = \vec{x}.$$

Often, we are interested in determining if a given vector is in the span of some set of other vectors. In particular, a system of linear equations has a solution precisely when the rightmost column of the augmented matrix is in the span of the columns to the left of it. This means a system of linear equations is equivalent to a single vector equation.

2 Determine if $\vec{x}$ is in the span of the given vectors:

(a) $\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$; $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$

(b) $\vec{x} = \begin{bmatrix} 12 \\ 14 \end{bmatrix}$; $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(c) $\vec{x} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$; $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Under the Hood: The span of a collection of vectors is essentially the set of all vectors that can be constructed using the members of the collection as components. This means that if a vector is *not* in the span of the collection, it has some additional component that's different from everything in the collection.

Matrix-Vector Products

The Punch Line: We can use even more compact notation than vector equations by introducing matrices. This will allow us to study systems of linear equations by studying matrices.

Warm-Up: Write the following systems of linear equations as vector equations:

(a) The system with variables z_1 and z_2

$$\begin{aligned} z_1 + 2z_2 &= 6 \\ 2z_1 - 5z_2 &= 3. \end{aligned}$$

(b) The system with variables x, y, and z

$$\begin{aligned} x &= x_0 \\ y &= y_0 \\ z &= z_0. \end{aligned}$$

(c) The system with variables x_1, x_2, and x_3

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 - x_3 &= 0. \end{aligned}$$

The Technique: The linear combination $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$ is represented by the matrix-vector product

$$A\vec{x} = \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}.$$

This means that to compute a matrix-vector product, we can just write it back out as a linear combination of the columns of the matrix. This means that matrix-vector products only work when there are precisely as many columns in the matrix as there are entries in the vector.

1 Compute the following matrix-vector products:

(a) $\begin{bmatrix}1 & 2\\2 & -5\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix}$

(b) $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$

(c) $\begin{bmatrix}1 & 1\\1 & -1\\2 & 0\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}$

(d) $\begin{bmatrix}1 & 2 & 3\\1 & 4 & 9\end{bmatrix}\begin{bmatrix}-1\\0\\2\end{bmatrix}$

Applications: The matrix equation $A\vec{x} = \vec{b}$ can be rephrased as the assertion that $\vec{b}$ is in the span of the columns of A. This gives us a geometric interpretation of systems of linear equations when we write them in matrix form—an equation being true means a particular vector, $\vec{b}$, is in the span of the collection of vectors $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ that make up the matrix A. In this case, the vector $\vec{x}$ is the collection of weights in a linear combination that proves $\vec{b}$ is in the span of the columns of A.

2 If possible, find at least one solution to each of these matrix equations (if not, explain why it is impossible):

(a) $\begin{bmatrix} 1 & 2 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

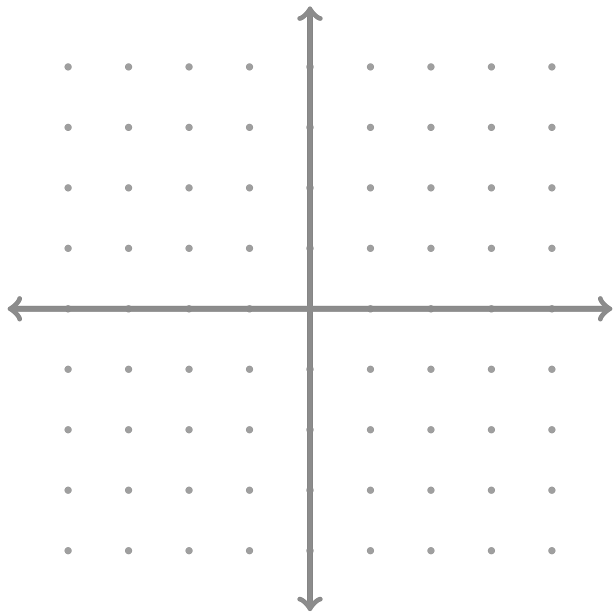
(d) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Under the Hood: Given any vector $\vec{b}$, the equation $A\vec{x} = \vec{b}$ means that $\vec{b}$ is in the span of the columns of A. This means that the span of the columns of A is related to the set of all possible matrix equations that could be solved with $A\vec{x}$ as the left hand side—there's one for each $\vec{b}$ in the span!

Solution Sets of Linear Systems

The Punch Line: There is a geometric interpretation to the solution sets of systems 0f linear equations, which allows us to explicitly describe them with *parametric equations*.

Warm-Up: Draw the line in $\mathbb{R}^2$ defined by $y = 3 - 2x$.



Verify that $x(t) = 1 + t$ and $y(t) = 1 - 2t$ satisfy the equation $y(t) = 3 - 2x(t)$ for all t, and plot $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ for $t = -1$, 0, and 1.

Homogeneous Equations: A matrix equation of the form $A\vec{x} = \vec{0}$ is called *homogeneous*. It always has the solution $\vec{x} = \vec{0}$, which is called the *trivial solution*. Any other solution is called a *nontrivial solution*; nontrivial solutions arise precisely when there is at least one free variable in the equation.

If there are m free variables in the homogeneous equation, the solution set can be expressed as the span of m vectors:

$$\vec{x} = s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_m\vec{v}_m.$$

This is called a *parametric equation* or a *parametric vector form* of the solution. A common parametric vector form uses the free variables as the parameters s_1 through s_m.

1 Find a parametric vector form for the solution set of the equation $A\vec{x} = \vec{0}$ for the following matrices A:

(a) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ 4 & -4 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & -2 & 0 \\ -2 & 0 & 4 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Nonhomogeneous Equations: A matrix equation of the form $A\vec{x} = \vec{b}$ where $\vec{b} \neq \vec{0}$ is called *nonhomogeneous*. As we've seen, a nonhomogeneous system may be inconsistent and fail to have solutions. If it does have a solution, though, we can find a parametric form for them as well as in the homogeneous case. Here, we express the solutions as $\vec{x} = \vec{p} + \vec{v}_h$, where $\vec{p}$ is some particular solution to the nonhomogeneous system (which we can get by picking simple values for the parameters, such as taking all free variables to be zero), and $\vec{v}_h$ is a parametric form for the solution to the *homogeneous* equation $A\vec{v}_h = \vec{0}$.

2 If possible, find a parametric vector form for the solution set of the nonhomogeneous equation $A\vec{x} = \vec{b}$ for the following matrices A and vectors $\vec{b}$ (otherwise explain why it is impossible):

(a) $\begin{bmatrix} 1 & 2 \end{bmatrix}; \begin{bmatrix} 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}; \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}; \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}; \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Under the Hood: Why do the solution sets to nonhomogeneous solutions have a "homogeneous part"? Imagine we are given two vectors, $\vec{x}_1$ and $\vec{x}_2$, and we're assured that $A\vec{x}_1 = \vec{b}$ and $A\vec{x}_2 = \vec{b}$. That is, we have two solutions to the nonhomogeneous equation. We can take the difference between these two equations to see that $A\vec{x}_1 - A\vec{x}_2 = \vec{b} - \vec{b}$. A property of matrix-vector multiplication lets us write the left-hand side as $A(\vec{x}_1 - \vec{x}_2)$, while the right-hand side is clearly $\vec{0}$, so we're left with the equation $A(\vec{x}_1 - \vec{x}_2) = \vec{0}$. That is, we've just shown the *difference* between two solutions to the nonhomogeneous equation is always a solution to the homogeneous equation with the same matrix!

Applications of Linear Systems

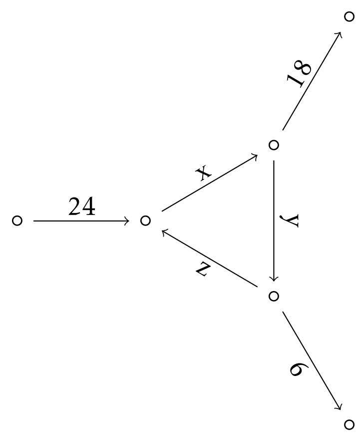
The Punch Line: Linear systems of equations can describe many interesting situations.

Set-Up: In a situation you can model with linear equations, there will be a number of *constraints*: things which must be equal because of the laws governing what's going on (e.g., laws of physics, economic principles, or definitions of quantities and the values you observe for them). These will give you the equations that you can solve to get information about the variables you care about

1 In the past three men's soccer games, the Gauchos averaged $\frac{5}{3}$ goals per game. They scored the same number of goals in the most recent two games, but three games ago they scored an additional two goals. How many points did they score in each game?

(This problem was actually true as of the 11^{th}, but even if you know the scores, it's probably helpful to set up the system and see how they come out of the equations.)

2 Suppose you're watching a bike loop on campus and writing down the net number of bicycles travelling through each part of the loop (the number of bikes going one direction minus the number going the other direction). You're able to observe how many net bikes per minute enter and leave through each of the three spokes, but aren't able to count well inside the loop. Luckily, you can use linear algebra to learn about how many net bikes per minute travel through each part of the loop (which is to say, find all solutions for x, y, and z that are consistent with the rest of the information about the problem)!



3 (Example 1 in Section 1.6) In an *exchange model* of economics, an economy is divided into different sectors which depend on each others' products to produce their output. Suppose we know for each sector its total output for one year and exactly how this output is divided or "exchanged" among the other sectors of the economy. The total dollar (or other monetary unit) value of each sector's output is called the *price* of that output. There is an *equilibrium price* for this kind of model, where each sectors income exactly balances its expenses. We wish to find this equilibrium.

Suppose we have an economy described by the following table:

Distribution of output from:			
Coal	**Electric**	**Steel**	**Purchased by:**
0.0	0.4	0.6	Coal
0.6	0.1	0.2	Electric
0.4	0.5	0.2	Steel

If we denote the price of the total annual outputs of the Coal, Electric, and Steel sectors by p_C, p_E, and p_S respectively, what is the equilibrium price (or describe them if there is more than one).

Under the Hood: When can we use linear equations to model something? The basic setup of a linear system involves a collection of quantities that we know are equal to known values (or each other), and a collection of variables. We can use a linear system when the way the quantities depend on changes to the variables is independent of the actual values of the variables (adding the same amount to a variable changes each quantity in the same way, no matter what value any of the variables have).

Linear Independence

The Punch Line: *Linear independence* is a property describing a collection of vectors whose span is "as big as it can be."

Warm-Up: Are each of these situations possible?

(a) The vectors $\{\vec{u}, \vec{v}\}$ in $\mathbb{R}^2$ span all of $\mathbb{R}^2$.

(b) The vectors $\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$ span all of $\mathbb{R}^3$.

(c) The vectors $\{\vec{x}, -\vec{x}\}$ span all of $\mathbb{R}^2$.

(d) The vectors $\{\vec{u}, \vec{v}\}$ span all of $\mathbb{R}^3$.

(e) The span of $\{\vec{u}_1, \vec{u}_2\}$ and the span of $\{\vec{v}_1, \vec{v}_2\}$ in $\mathbb{R}^3$ intersect only at $\vec{0}$.

(f) The span of $\{\vec{u}_1, \vec{u}_2\}$ and the span of $\{\vec{v}_1, \vec{v}_2\}$ in $\mathbb{R}^3$ are both planes and intersect only at $\vec{0}$.

Tests for Linear Independence: That the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent means that if $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$, the only possibility is that $c_1 = c_2 = \cdots = c_n = 0$. That is, the only solution to the homogeneous matrix equation

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$$

is the trivial solution: the zero vector. If there's only one vector in the set, it is linearly independent unless it happens to be $\vec{0}$. If there's only two vectors, they are linearly independent unless one is a multiple of the other (including $\vec{0}$, which is 0 times any vector). If a subset of the vectors is linearly dependent, the whole set is. Finally, a set of vectors is linearly dependent if (and only if) at least one vector is in the span of the others.

1 Are these sets linearly dependent or independent? Why?

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{3}{4} \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 12 \\ 1 \end{bmatrix} \right\}$

2 Are each of these situations possible?

(a) You have a set of vectors that spans $\mathbb{R}^3$. You remove two of them, and the set of vectors left behind is linearly dependent.

(b) You have two sets of vectors in $\mathbb{R}^6$. One has four vectors, and one has two vectors, and both sets are linearly independent. When you put both sets together, the resulting set of six vectors is linearly dependent.

(c) You have a set of three vectors which span $\mathbb{R}^3$, but it is linearly dependent.

(d) You have a linearly dependent set of three vectors in $\mathbb{R}^2$. If you remove any one of them, the other pair do not span $\mathbb{R}^2$.

Linear Transformations

The Punch Line: Matrix multiplication defines a special kind of function, known as a *linear transformation*.

Warm-Up: What do each of these situations mean (geometrically, algebraically, in an application, and/or otherwise)?

(a) The product of the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

(b) The vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(c) The equation $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has a solution.

(d) The set of vectors $\vec{x}$ such that the matrix equation $A\vec{x} = \vec{b}$ is satisfied forms a plane in $\mathbb{R}^3$.

(e) The set of vectors $\vec{b}$ such that the matrix equation $A\vec{x} = \vec{b}$ is satisfied forms a line in $\mathbb{R}^2$.

(f) For two particular vectors $\vec{x}$ and $\vec{b}$, and a matrix A, the matrix equation $A\vec{x} = \vec{b}$ is satisfied.

What They Are: A *linear transformation* is a mapping T that obeys two rules:

(a) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}$ and $\vec{v}$ in its domain,

(b) $T(c\vec{u}) = cT(\vec{u})$ for all scalars c and $\vec{u}$ in its domain.

These rules lead to the rule $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$ for c, d scalars and $\vec{u}, \vec{v}$ in the domain of T, and in fact $T(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_nT(\vec{v}_n)$. That is, the transformation of a linear combination of vectors is a linear combination of the transformations of the vectors (with the same coefficients).

1 Are each of these operations linear transformations? Why or why not?

(a) $T(\vec{x}) = 4\vec{x}$

(b) $T(\vec{x}) = A\vec{x}$ for some matrix A (with the right number of columns)

(c) $T(\vec{x}) = \vec{0}$

(d) $T(\vec{x}) = \vec{b}$ for some nonzero $\vec{b}$

(e) $T(\vec{x}) = \vec{x} + \vec{b}$ for some nonzero $\vec{b}$

(f) $T(\vec{x})$ takes a vector in $\mathbb{R}^2$ and rotates it by 45° ($\frac{\pi}{4}$ radians) counter-clockwise in the plane

What They Do: Linear transformations convert between two different spaces, such as $\mathbb{R}^n$ and $\mathbb{R}^m$. If $n = m$, then we can also think of them moving around the vectors inside $\mathbb{R}^n$ (e.g., by rotation or stretching).

2 What do the linear transformations corresponding to multiplication by these matrices do, geometrically? (Try applying the matrix to a vector composed of variables, then examining the result, or multiplying by a few simple vectors and sketching what happens.)

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

The Matrix of a Linear Transformation

The Punch Line: Linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ are *all* equivalent to matrix transformations, even when they are described in other ways.

Warm-Up: What does the linear transformation corresponding to multiplication by each of these matrices do geometrically (don't worry too much about the exact values for things like rotation or scaling)?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Getting the Matrix: We can write down a matrix that accomplishes any linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ by writing down what the transformation does to the vectors corresponding to each component (these have a single 1 and the rest of their entries as zeros, and make up the columns of the $n \times n$ *identity matrix*, which has ones down the diagonal and zeros elsewhere).

1 Write down a matrix for each of these linear transformations.

(a) In $\mathbb{R}^2$, rotation by 180° (π radians) counterclockwise.

(b) In $\mathbb{R}^3$, rotation by 180° (π radians) counterclockwise in the xz plane.

(c) In $\mathbb{R}^2$, stretching the x direction by a factor of 2 then reflecting about the line $y = x$.

(d) In $\mathbb{R}^3$, the transformation that looks like a "vertical" (that is, the z direction is the one which moves) shear in both the xz and yz planes, each with a "shear factor" (the amount the corner of the unit square moves) of 2.

[Note: Don't worry too much if this one's harder than the rest, shear transformations are hard to describe. If you get stuck, it might be a good idea to work on Problem 2 rather than sink in too much time here.]

One to One and Onto: When describing a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$, we say T is *one to one* if each vector in $\mathbb{R}^m$ is the image of at most one vector in $\mathbb{R}^n$ (it can fail to be the image of any vector, it just can't be the image of two different ones). We say T is *onto* if each vector in $\mathbb{R}^m$ is the image of at least one vector in $\mathbb{R}^n$ (it can be the image of more than one).

We can test these conditions with ideas we already know: T is one-to-one if and only if the columns of its matrix are linearly independent, and onto if and only if they span $\mathbb{R}^m$. An equivalent test for T being one-to-one is that the equation $A\vec{x} = \vec{0}$ (where A is the matrix of T) has only the trivial solution if and only if T is one-to-one. An equivalent test for onto is that $A\vec{x} = \vec{b}$ is consistent for all $\vec{b}$ in $\mathbb{R}^m$.

2 Determine if the linear transformations with the following matrices are one-to-one, onto, both, or neither.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 2 & 1 & 0 \\ 6 & -3 & 12 \\ 5 & 2 & 1 \end{bmatrix}$

Why does the $A\vec{x} = \vec{0}$ test work? If $A\vec{x} = A\vec{y}$, then $A(\vec{x} - \vec{y}) = \vec{0}$. If x and y weren't the same to begin with, then their difference is mapped to $\vec{0}$ by A as a consequence of them having the same value for the product. Similarly, if $A\vec{z} = \vec{0}$ for a nonzero $\vec{z}$, then $A(\vec{x} + \vec{z}) = A\vec{x} + A\vec{z} = A\vec{x}$, even though $\vec{x} \neq \vec{x} + \vec{z}$.

Matrix Operations

The Punch Line: Various operations combining linear transformations can be realized with some standard matrix operations.

Addition and Scalar Multiplication: Just like with vector operations, the sum of matrices and the multiplication by a *scalar* (just a number, as opposed to a vector or matrix) are done component-by-component.

1 Try the following matrix operations:

(a) $3\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Matrix Multiplication: To multiply two matrices, we create a new matrix, each of whose columns is the result of the matrix-vector product of the left matrix with the corresponding column of the right matrix (the product will have the same number of rows as the left matrix, and the same number of columns as the right matrix). To get the ij entry (ith row and jth column) we could multiply the ith row of the left matrix with the jth column of the right matrix.

2 Multiply these matrices (if possible, otherwise say why it isn't):

(a) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$

Transpose: The last matrix operation for today is the *transpose*, where you switch the roles of rows and columns. That is, if you get an $n \times m$ matrix, its transpose will be $m \times n$.

3 Compute the following operations for the matrices given:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}.$$

(a) A^T

(b) B^T

(c) C^T

(d) $(BA)^T$

(e) $A^T B^T$

(f) $(BAC)^T$

(g) AA^T

(h) $A^T A$

(i) $(AA^T - B)^T$

What do these operations mean? Matrix addition and scalar multiplication correspond to adding and scaling the results of applying the linear transformation of the matrix, respectively. Matrix multiplication corresponds to composing the two linear transformations (applying one to the result of another). Transposition is a little weirder, and corresponds to switching the roles of variables and coefficients in a linear equation.

The Inverse of a Matrix

The Punch Line: Undoing a linear transformation given by a matrix corresponds to a particular matrix operation known as *inverse*.

Warm-Up: Are the following vector operations reversible/invertible?

(a) $T(\vec{x}) = 4\vec{x}$

(b) $T(\vec{x})$ is counterclockwise rotation in the plane by 45° ($\frac{\pi}{4}$ radians)

(c) $T(\vec{x}) = \vec{0}$

(d) $T(\vec{x}) = \vec{x} + \vec{b}$

(e) $T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\vec{x}$

(f) $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\vec{x}$

The Inverse: The *inverse* of an $n \times n$ matrix A is another matrix B that satisfies the two matrix equations $AB = I_n$ and $BA = I_n$, where the *identity matrix* I_n has ones on the diagonal and zeroes everywhere else. We use the notation A^{-1} to refer to such a B (which, if it exists, is unique).

We can find the inverse of a matrix by applying row operations to the augmented matrix $\begin{bmatrix} A & I_n \end{bmatrix}$ (which is augmented with the n columns of the identity matrix, rather than a single vector). If the left part of the augmented matrix can be transformed by row operations to I_n, then the right part will be transformed by those row operations to A^{-1}. If the system is inconsistent, the matrix A is not invertible (and we may call it *singular*).

1 Find the inverse of these matrices (you may want to check your results by multiplying the result with the original matrix):

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & -1 \\ 7 & -2 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Relevance to Matrix Equations: The inverse of a matrix allows you to "reverse engineer" a matrix equation, in the sense that if $A\vec{x} = \vec{b}$ and A is invertible, then $\vec{x} = A^{-1}\vec{b}$ is a solution to the original equation. In fact, it is the unique solution to the equation!

2 Use the inverses computed previously to solve these matrix equations:

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & -1 \\ 7 & -2 \end{bmatrix}\vec{x} = \begin{bmatrix} a \\ a+1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Computing the inverse of a matrix reveals the structure of how to invert the linear transformation it represents. As the book notes, it can be faster to simply perform row operations to find a solution to any particular matrix equation. However, looking at the inverse matrix can give a more geometric idea of what undoing some particular operation is—to undo a rotation and shear requiring a different shear and rotation in the opposite direction, for example.

Characterizing Invertible Matrices

The Punch Line: There are many equivalent conditions to determine if a matrix is invertible, and describe properties of ones that we know are invertible.

Warm-Up: How big is the solution set of the homogeneous equation with these matrices (is it finite or infinite? what is its dimension?)? How about the span of their columns?

(a) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$

Matrix Conditions: Our first definition for invertible matrices states that A is invertible if some other matrix C makes the equations $AC = I_n$ and $CA = I_n$ simultaneously true. We can show that if A is invertible, so is A^T, that the inverse of A^T is C^T, and that if either one of the two equations in the definition is true, the other one must be as well, by playing around with transposing the equations.

1 Can each of these things happen? Do they have to be true?

(a) A is invertible and the matrix C with $AC = I_n$ is also invertible

(b) C is an inverse to both A and A^T (that is, $CA = I_n$ and $CA^T = I_n$)

(c) $CA = I_n$ and $AD = I_n$, but $C \neq D$

(d) $ABCD = I_n$, but $AB \neq I_n$ and $CD \neq I_n$

Equation Conditions: We also have conditions based on the homogeneous and inhomogeneous equations involving the matrix. We know A is invertible if it is square ($n \times n$) and its columns span $\mathbb{R}^n$ or are linearly independent (for square matrices these are equivalent, though not in general). That is, it has n pivots (so its EF's have pivots in every column, and its REF is I_n) or no free variables. That is, the equation $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$ (which will turn out to be unique) or $A\vec{x} = \vec{0}$ has *only* the trivial solution.

2 Are these matrices invertible?

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 2 \\ -1 & 1 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 3 & 1 & -4 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Why do the columns have to be linearly independent and span all of $\mathbb{R}^n$? If they were not linearly independent, there would be multiple solutions to $A\vec{x} = \vec{b}$, so we couldn't define $A^{-1}\vec{b} = \vec{x}$—we wouldn't know which to choose! And if they did not span $\mathbb{R}^n$, then there would be some $\vec{b}$ outside their span where we couldn't find any $\vec{x}$ so that $A\vec{x} = \vec{b}$—we'd again have a problem defining the inverse, but this time instead of having to many possible answers $\vec{x}$, we wouldn't have any!

Determinants!

The Punch Line: We can compute a value from the entries of a matrix to get yet *another* way of characterizing invertible matrices. ***SPOILER ALERT***: The determinant will also give us a variety of other useful pieces of information in understanding a matrix and its associated linear transformation!

Warm-Up: Are these matrices invertible? Are there conditions that make them so or not so depending on certain values? Try to answer without reducing them to REF (and in general, with as few computations as possible).

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

(d) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & -1 \\ 1 & 1 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix}$

The Definition: We define the *determinant* of a matrix in general to be

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}).$$

In this definition, we're moving along the first row, taking (-1) to be one power higher than the column we're in (this means take a positive value for odd columns and a negative one for even columns), multiplying by the entry we find, then taking the determinant of the smaller matrix obtained by ignoring the top row and the column we're in. This involves the determinant of smaller matrices, so if we drill down enough layers, we'll get back to 2×2 matrices, where we can just use the formula $ad - bc$ from earlier.

1 Find the determinants for each of these matrices:

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 4 \\ 1 & -1 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ -1 & -2 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{bmatrix}$

Cofactor Expansion: We can actually expand along *any* row or column. In that case, the (-1) has exponent $(-1)^{i+j}$ (where i marks the row and j the column we're in), the matrix entry is a_{ij}, and the subdeterminant is $\det(A_{ij})$. The goal here is to find the simplest row or column to move along to minimize the amount of computation. Mostly, this means finding the row or column with the most zeros, and the "nicest" (e.g., smallest) nonzero entries.

2 Try to compute the determinants of the following matrices by computing as few subdeterminants as possible.

(a) $\begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & -1 \\ -2 & 0 & 3 \end{bmatrix}$ (min is 1 subdeterminant)

(b) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 4 \\ 1 & 0 & -1 \end{bmatrix}$ (min is 2 subdeterminants)

(c) $\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ (min is 3 subdeterminants if you use a result from problem 1, and 4 otherwise)

(d) $\begin{bmatrix} 2 & -1 & 0 & 3 \\ 7 & -2 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ -3 & -2 & 0 & 1 \end{bmatrix}$ (min is 2 subdeterminants)

Determinants II: Return of Row Operations

The Punch Line: We can use row operations to calculate determinants if we're careful.

The Process: Our three row operations—interchange, scaling, and replacement with a sum—have predictable effects on the determinant. By tracking the operations we use to get a matrix that is easy to compute a determinant for—generally a matrix in echelon form (which is also in triangular form)—we can avoid most of the work involved. In particular, interchanging two rows multiplies the determinant by -1, scaling a row by k scales the determinant by k, and replacing a row with its sum with a multiple of a *different* row does not change the determinant. In practice, we mostly want to interchange rows and use scaled sums to get to Echelon Form (not necessarily reduced!), then multiply the diagonal entries.

1 Use row operations to help compute these determinants:

(a) $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 1 & -1 \end{vmatrix}$

(c) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix}$

(d) $\begin{vmatrix} 1 & 2 & 3 & 0 & 4 \\ 12 & \sqrt{\pi} & e^{e^e} & 1 & \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{2}}}} \\ 0 & 0 & 2 & 0 & 6 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{2}{3} \\ 0 & \frac{1}{2} & 0 & 0 & 1 \end{vmatrix}$

Column Operations and Other Properties: Since $\det(A) = \det(A^T)$ (which takes a bit of argument to show), we can also do column operations analogous to the row operations, with the same effect on the determinant. Interspersing them can be helpful. Another useful property is that $\det(AB) = \det(A)\det(B)$ (although $\det(A+B)$ is often not $\det(A)+\det(B)$).

2 Find expressions for the following determinants (and justify them):

(a) $\det\left(A^2\right)$

(b) $\det(A^n)$

(c) $\det(BA)$

(d) $\det\left(A^{-1}\right)$

(e) $\det(kA)$ (where k is some real number)

(f) $\begin{vmatrix} A & O \\ O & B \end{vmatrix}$

In the last problem, A and B are standing for the entries of matrices A and B filling out those portions of the matrix, and O stands for zeros in those entries (so if A is $n\times n$ and B is $m\times m$, this matrix is $(n+m)\times(n+m)$. This is something of a challenge problem—I expect it's more abstract than most problems you'll be given.

Vector Spaces

The Punch Line: The same ideas we've been using translate to work on more abstract vector spaces, which describe many things which occur in "nature" (at least, in the mathematics we use to describe nature).

The Rules: A *vector space* is a set of objects V that satisfy these 10 axioms:

1. $\vec{x} + \vec{y} \in V$ (we say V is closed under addition)
2. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (addition is commutative)
3. $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$ (addition is associative)
4. There is a $\vec{0}$ with the property that $\vec{x} + \vec{0} = \vec{x} = \vec{0} + \vec{x}$
5. There is a $\overrightarrow{-x}$ for every $\vec{x}$ so $\vec{x} + \overrightarrow{-x} = \vec{0}$
6. $c\vec{x} \in V$ (V is closed under scalar multiplication)
7. $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$ (left distributivity)
8. $(c+d)\vec{x} = c\vec{x} + d\vec{x}$ (right distributivity)
9. $c(d\vec{x}) = (cd)\vec{x}$ (scaling is associative)
10. $1\vec{x} = \vec{x}$ (multiplicative identity)

1 Are these things vector spaces?

(a) The subset $\left\{\vec{0}\right\}$ in any $\mathbb{R}^n$

(b) $\mathbb{R}^2$ but scalar multiplication $c\vec{x}$ is defined as $c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x/c \\ y/c \end{bmatrix}$.

(c) $\mathbb{R}^2$ but addition is defined as $\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} y+w \\ x+z \end{bmatrix}$

(d) All functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$

(e) The set of all functions of the form $f(\theta) = a\sin(\theta + \phi)$

(f) The set of vectors $\vec{b}$ such that $\vec{b} = A\vec{x}$ (where A is fixed)

Subspaces: A subset U of a vector space V is a *subspace* if it contains $\vec{0}$ and is closed under addition and scaling. A subspace is a vector space in its own right.

2 Are these subsets subspaces?

(a) The vectors in $\mathbb{R}^3$ whose entries sum to zero

(b) The vectors in $\mathbb{R}^2$ which lie on one of the axes

(c) The vectors in $\mathbb{R}^3$ that are mapped to zero by matrices A and B

(d) The functions of the form $f(\theta) = A\sin(\theta + \phi)$ with ϕ rational

(e) The functions of the form $f(\theta) = A\sin(\theta + \phi)$ with ϕ irrational

(f) The functions of the form $f(\theta) = A\sin(\theta + \phi)$ with A rational

Why do we want sets to be vector spaces? In some sense, vector spaces all work in the same way, so if we can show that some set we're interested in is a vector space, we get to import all kinds of results "for free." We're taking something we know how to work with—$\mathbb{R}^n$—and leveraging it to get answers to things that are harder to deal with—like differential equations (see future math courses).

Column and Null Spaces

The Punch Line: The sets of vectors we've been most interested in so far in the course—solution sets (to homogeneous systems) and spans—are in fact subspaces!

Warm-Up: Can these situations happen?

(a) A vector $\vec{x}$ is in both the null space and column space of a 3×5 matrix

(b) A vector $\vec{x}$ is in both the null space and column space of a 2×2 matrix

(c) A vector $\vec{x}$ is in neither the null space nor column space of a 2×2 matrix

(d) A vector $\vec{x}$ is in neither the null space nor column space of an invertible 4×4 matrix

Null Spaces: The *null space* (also called the *kernel*) of a linear transformation T in the vector space V is the set of all vectors $\vec{x}$ that are mapped to $\vec{0} \in V$ by T: $T(\vec{x}) = \vec{0}$. For $\mathbb{R}^n$ and $T(\vec{x}) = A\vec{x}$ for a matrix A, we can explicitly describe the vectors in the null space by finding a parametric form for the solution set of the homogeneous equation $A\vec{x} = \vec{0}$. The vectors attached to each parameter span the null space.

1 Describe the null spaces of the following linear transformations:

(a) $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\vec{x}$

(b) $T(\vec{x}) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 1 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix}\vec{x}$

(c) $T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\vec{x}$

(d) $T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\vec{x}$

(e) $T(\vec{x}) = \begin{bmatrix} 8 & 6 & 7 & 5 \\ 3 & 0 & 9 & 9 \end{bmatrix}\vec{x}$

(f) $T(\vec{x}) = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -1 & 2 \\ 0 & 0 \end{bmatrix}\vec{x}$

(g) $T(f(x)) = f(x) - f(0)$ acting on the space of all $\mathbb{R} \to \mathbb{R}$ functions*

*This is something of a challenge problem; it should help you understand null spaces, but it probably won't be on an exam.

Column Spaces and Range: The *column space* of a matrix is the span of its columns. For more general linear transformations, the analogous concept is *range*—the set of vectors in the vector space V that can be reached by applying the linear transformation. In $\mathbb{R}^n$, we can get the column space as just the span of the columns (although we can describe it more succinctly if we eliminate linearly dependent columns).

2 Describe the range of these linear transformations. What is their dimension? Try to find a spanning set with only that many vectors. See if you can relate these situations to the null spaces you found on the last page.

(a) $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}$

(b) $T(\vec{x}) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 1 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 \end{bmatrix} \vec{x}$

(c) $T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}$

(d) $T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}$

(e) $T(\vec{x}) = \begin{bmatrix} 8 & 6 & 7 & 5 \\ 3 & 0 & 9 & 9 \end{bmatrix} \vec{x}$

(f) $T(\vec{x}) = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -1 & 2 \\ 0 & 0 \end{bmatrix} \vec{x}$

(g) $T(f(x)) = f(x) - f(0)$ acting on the space of all $\mathbb{R} \to \mathbb{R}$ functions*

*This is again a challenge problem. What could the dimension be here?

What's going on with the linear transformation in part (d)? When (part of) the column space is in the null space, the matrix is sending vectors somewhere it will send to zero. If we applied the transformation twice (or, in general, enough times), it would send all vectors to zero. It's kind of a drawn-out process: send vectors matching some description (in some span) to zero, then change other vectors to take their places. It's important to remember that the null space is describing where vectors are *before* the transformation, while the column space is describing *after*.

Bases

The Punch Line: We have an efficient way to define subspaces using collections of vectors in them.

Warm-Up: Are these sets linearly independent? What do they span?

(a) $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\2\end{bmatrix},\begin{bmatrix}1\\2\\2\end{bmatrix}\right\} \subset \mathbb{R}^3$

(b) All vectors in $\mathbb{R}^{42}$ with a zero in at least one component

(c) $\left\{\begin{bmatrix}2\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\-1\\1\end{bmatrix},\begin{bmatrix}-2\\2\\-3\\2\end{bmatrix}\right\} \subset \mathbb{R}^4$

(d) $\left\{1, t-1, (t-1)^2 + 2(t-1)\right\} \subset \mathscr{P}_2$

Bases: A *basis* for a vector space is a linearly independent spanning set. Every finite spanning set contains a basis by removing linearly dependent vectors, and many finite linearly independent sets may be extended to be a basis by adding vectors (if eventually this process terminates in a spanning set).

1 Are these sets bases for the indicated vector spaces? If not, can vectors be removed (which?) or added (how many?) to make it a basis?

(a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\} \subset \mathbb{R}^3$

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\} \subset \mathbb{R}^2$

(c) $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ -3 \\ 2 \end{bmatrix} \right\} \subset \mathbb{R}^4$

(d) $\left\{ (t-1), (t-1)^2, (t-1)^3 \right\} \subset \mathscr{P}_3$

Finding Bases in $\mathbb{R}^n$: We're often interested in subspaces of the form Nul A and Col A for some matrix A. Fortunately, we can extract both by examining the Reduced Echelon Form of A.

A basis for Col A consists of all columns in A itself which correspond to pivot columns in the REF of A. A basis for Nul A consists of the vector parts corresponding to each free variable in a parametric vector representation of the solution set of the homogeneous equation $A\vec{x} = \vec{0}$, which we can find from the REF of A. Caution: In general, although free variables correspond to non-pivot columns in the REF, the basis for Nul A will *not* consist of those columns—in fact, they will often be of the wrong size!

2 Find bases for Nul A and Col A for each matrix below:

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a \neq 0$

Coordinates

The Punch Line: If we have a basis of n vectors for any vector space, we can describe (and work with) any vector from the space or equation in it as if it were in $\mathbb{R}^n$ all along!

Coordinate Vectors: If we have an *ordered* basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for vector space V, then any vector $v \in V$ has a unique representation

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n,$$

where each c_i is a real number. Then we can write the *coordinate vector* $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$.

1 Find the representation of the given vector $\vec{v}$ with respect to the ordered basis $\mathcal{B}$.

(a) $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, $\vec{v} = \begin{bmatrix} 8 \\ 0 \\ 5 \end{bmatrix}$

(b) $\mathcal{B} = \left\{1, t, t^2, t^3\right\}$, $\vec{v} = t^3 - 2t^2 + t$

(c) $\mathcal{B} = \left\{1, (t-1), (t-1)^2, (t-1)^3\right\}$, $\vec{v} = t^3 - 2t^2 + t$

(d) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

(e) $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$,
$\vec{v} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

Change of Coordinates in $\mathbb{R}^n$: If we have a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$, we can recover the standard representation by using the matrix P whose columns are the (ordered) basis elements represented in the standard basis:

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}.$$

The matrix P^{-1} takes vectors in the standard encoding and represents them with respect to $\mathcal{B}$. Thus, if $\mathcal{C}$ is another basis for the same space and Q is the matrix bringing representations with respect to $\mathcal{C}$ to the standard basis, then $Q^{-1}P$ is a matrix which takes a vector encoded with respect to $\mathcal{B}$ and returns its encoding with respect to $\mathcal{C}$. That is,

$$[\vec{v}]_{\mathcal{C}} = Q^{-1} P \, [\vec{v}]_{\mathcal{B}}.$$

2 Compute the change of basis matrices for the following bases (into and from the standard basis).

(a) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ (d) $\left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$

3 Compute the change of basis matrices between the two bases:

(a) $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(b) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}, \mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$

Dimension and Rank

The Punch Line: We can compare the "size" of different vector spaces and subspaces by looking at the size of their bases.

Warm-Up: Are these bases for the given vector space?

(a) $\left\{\begin{bmatrix}2\\5\end{bmatrix}, \begin{bmatrix}5\\-2\end{bmatrix}\right\}$ in $\mathbb{R}^2$

(b) $\{1, 1+t, t\}$ in $\mathscr{P}_1$

(c) $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right\}$ in the vector space (you can check it is one) of all 2×2 matrices

(d) $\left\{1, t, t^2, \ldots, t^n\right\}$ for some fixed n in the space of all polynomials

Dimension: If one basis for a vector space V has n vectors, then all others do. We can see this by writing the other basis' coordinates with respect to the first basis, then looking at the Reduced Echelon Form of this matrix—there can't be any free variables, and there must be n pivots, so there must be n vectors in the new basis.

1 Find the dimension for each of the following subspaces.

(a) $\text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

(b) $\text{Span}\left\{1-t+t^2, 1+t-2t^2, t^2-t, t^3-t, t^3-t^2\right\}$

(c) $\text{Nul}\begin{bmatrix}1 & 1 & 1\\1 & -1 & 1\\-1 & 1 & -1\end{bmatrix}$

(d) $\text{Col}\begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\-1 & 0 & 1\end{bmatrix}$

Rank of a Matrix: For any $n \times m$ matrix A, the dimension of the null space is the number of free variables and the dimension of the column space is the number of pivots. These add up to m, the number of columns (a column is either a pivot or corresponds to a free variable). We call the dimension of the column space the *rank* of a matrix.

2 Find the ranks of the following matrices:

(a) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -2 \\ 1 & 0 & 0 & 6 \end{bmatrix}$

(d) An invertible $n \times n$ matrix

The statement that $\operatorname{rank} A + \dim \operatorname{Nul} A$ is the number of columns of A is an important theorem known as the Rank Nullity Theorem (some people call $\dim \operatorname{Nul} A$ the *nullity* of A). It is basically saying that the input space to A has only two important parts: the null space, and the vectors which contain the information for knowing what the column space looks like. There's a bit more to it than that, but the gist is there isn't some third kind of vector lurking around that isn't related to either the null or column spaces.

Eigenvalues

The Punch Line: If we have a linear transformation from an n-dimensional vector space to itself, we can choose a basis that makes the matrix of the linear transformation especially simple—characterized by just n constants.

Warm-Up: Are the following matrices invertible? If not, what is the dimension of their null space?

(a) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}$

(e) $\begin{bmatrix} -1 & 2 & 4 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Eigenvalues and Eigenvectors: If V is an n-dimensional vector space and T is a linear transformation from V back into itself, and we find a (nonzero) vector $\vec{v} \in V$ and $\lambda \in \mathbb{R}$ that make the equation $T(\vec{v}) = \lambda\vec{v}$, we call λ and *eigenvalue* and $\vec{v}$ an *eigenvector* for T. The eigenvectors of the linear transformation are vectors whose direction does not change when you apply the transformation (except possibly reversing if $\lambda < 0$). The eigenvalues of the linear transformation are the different "scaling factors" that the transformation uses (as well as containing information about whether it reverses direction of certain vectors).

1 Are these vectors eigenvectors of the given linear transformation? If so, what are their eigenvalues?

(a) $\vec{v} = t^2 \in \mathscr{P}_2$ with $T(p(t)) = t\frac{d}{dt}[p(t)]$

(b) $\vec{v} = t \in \mathscr{P}_2$ with $T(p(t)) = t\frac{d}{dt}[p(t)]$

(c) $\vec{v} = 1 \in \mathscr{P}_2$ with $T(p(t)) = t\frac{d}{dt}[p(t)]$

(d) $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\vec{x}$

(e) $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with $T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\vec{x}$

(f) $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ with $T(\vec{x}) = \begin{bmatrix} 1 & -5/3 \\ 0 & -3/2 \end{bmatrix}\vec{x}$

(g) $\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ with $T(\vec{x}) = \vec{0}$

Eigenspaces: The set of eigenvectors for eigenvalue λ of a given linear transformation is almost a subspace—all it's missing is the zero vector, which we may as well add in (after all, it also satisfies the equation $T(\vec{v}) = \lambda\vec{v}$). This means we can find a subspace corresponding to each eigenvalue of the linear transformation—we call it the *eigenspace for eigenvalue* λ, and denote it E_λ.

This is the null space of a linear transformation which is a slight modification of the original: if our transformation had matrix A, then E_λ is the null space of $A - \lambda I_n$. This is because if $(A - \lambda I_n)\vec{v} = 0$, then $A\vec{v} - \lambda I_n \vec{v} = A\vec{v} - \lambda\vec{v} = \vec{0}$, or $A\vec{v} = \lambda\vec{v}$. This means that if we know λ is an eigenvalue of the transformation, we can find its eigenspace by using techniques we already know for describing null spaces! We can also prove some number $\mu \in \mathbb{R}$ is *not* an eigenvalue by showing that its "eigenspace" (the null space of $A - \mu I_n$) is just $\left\{\vec{0}\right\}$, which isn't an eigenvector.

2 Determine if λ is an eigenvalue for the (transformation given by the) matrix A by computing E_λ:

(a) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\lambda = 1$

(b) $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$, $\lambda = 2$

(c) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $\lambda = 1$

(d) A is an invertible matrix, $\lambda = 0$

Under the Hood: Eigenspaces are a very interesting and important class of *invariant subspaces*—subspaces that are preserved by the transformation, in that any vector in the subspace will be mapped to another vector in the same subspace. The action of the transformation is very simple in the eigenspaces, so this is a huge win—if we can break any vector up into pieces that are all in eigenspaces, we can describe what happens to it just by seeing how each of those pieces gets scaled by the appropriate eigenvalue.

As it turns out, eigenspaces tell most of the story of invariant subspaces. There are only two other kinds in $\mathbb{R}^n$: spaces where the transformation looks like a rotation rather than a scaling, and spaces where the transformation "eventually" works like a scaling (after you apply it enough times). In this course, though, we'll just be focused on eigenspaces as they're presented here (ask me if you want to know more, or take Math 108).

Eigenvalues and Where to Find Them

The Punch Line: Finding the eigenvalues of a matrix boils down to finding the roots of a polynomial.

Warm-Up: What are the eigenvalues of these matrices? What is the dimension of each eigenspace?

[Note: you shouldn't have to do many computations here—just look at Echelon Forms and try to see how many free variables there will be.]

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 2 & 0 & 8 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

(e) $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$

(f) $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

The Characteristic Equation: If λ is an eigenvalue of the matrix A, that means there is some nonzero $\vec{v} \in \mathbb{R}^n$ that satisfies the equation $A\vec{v} = \lambda\vec{v}$. Then $(A - \lambda I_n)\vec{v} = \vec{0}$ (from putting all terms with $\vec{v}$ on the same side), so $(A - \lambda I_n)$ is a non-invertible matrix (it has nontrivial null space, because $\vec{v} \neq \vec{0}$). Since we know that a matrix being not invertible is equivalent to its determinant being zero, we can check when the equation $\det(A - \lambda I_n) = 0$ is true. This gives a polynomial equation in λ of degree n (why?), so if we can find the roots of the polynomial, we know all of the eigenvalues. This equation is known as the *characteristic equation*.

1 What is the characteristic equation for each of these matrices?
[Note: You need not solve the characteristic equation yet.]

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} -3/2 & 1/2 \\ 1/2 & -3/2 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$

2 What are the eigenvalues of these matrices? What are the dimensions of each eigenspace?

[Note: Again, try to minimize computation—we're not after the eigenspace itself, just its dimension, so you only need to manipulate the matrix into an Echelon Form matrix, not fully solve for its null space.]

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} -3/2 & 1/2 \\ 1/2 & -3/2 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$

Under the Hood: You may have noticed we often got one-dimensional eigenspaces. If we know something is an eigenvalue, we know that its eigenspace is *at least* one dimensional, and the eigenspaces for different eigenvalues are distinct except for the zero vector (otherwise A would act on a vector in both by scaling by the different eigenvalues, which would give two different answers!). Thus, if we have n distinct eigenvalues for a matrix in $\mathbb{R}^n$, we know we have found n distinct subspaces, each of which is at least one-dimensional. This means they *have* to be one-dimensional, otherwise $\mathbb{R}^n$ would have more than n dimensions! As it turns out, the characteristic equation gives us information on the maximum size of each eigenspace, through the *multiplicity* of each root (how many times it appears).

Diagonalization

The Punch Line: Eigenvalues and -vectors can be used to factor a matrix in a way that makes computation easier.

Warm-Up What are the eigenvalues of these matrices? What are their eigenspaces?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & -2 & 1 \\ -2 & 2 & -2 \\ 1 & -2 & -1 \end{bmatrix}$

Diagonalizing: If the matrix A has eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and eigenvectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ corresponding to them, then we write $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$ for the matrix whose columns are the eigenvectors and D for the matrix with the eigenvalues down the diagonal and zeroes elsewhere. Then

$$AP = A\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = PD.$$

If the eigenvectors are linearly independent, then P is invertible, and $A = PDP^{-1}$.

1 Are these matrices diagonalizable? If so, what are P and D?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & -2 & 1 \\ -2 & 2 & -2 \\ 1 & -2 & -1 \end{bmatrix}$

Using the Diagonalization: If we have written $A = PDP^{-1}$ with D a diagonal matrix, then we can easily compute the kth power of A as $A^k = PD^kP^{-1}$ (adjacent P and P^{-1} matrices will cancel, putting all of the D matrices together and just leaving the ones on the end).

2 The Fibonacci numbers are a *very* famous sequence of numbers. The first one is $F_1 = 0$, the second is $F_2 = 1$, and from then on out, each number is the sum of the previous two $F_n = F_{n-1} + F_{n-2}$ (this is sometimes used as a simple model for population growth—although it assumes immortality). Since it's annoying to compute F_n if n is very large (we'd have to do a lot of backtracking to get to known values), it would be nice to have a closed form for F_n. We can derive one with the linear algebra we already know!

(a) Since the equation defining F_n in terms of F_{n-1} and F_{n-2} is linear, we can use a matrix equation to represent the situation. In particular, we want a matrix A such that

$$A\begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix},$$

so that we can keep applying A to get further along in the sequence. What is this A?

(b) Since $\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = A\begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} = A^2\begin{bmatrix} F_{n-2} \\ F_{n-3} \end{bmatrix} = \cdots$, we can find F_n by computing $A^{n-2}\begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = A^{n-2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (we only need to advance by $n-2$ steps, because the top entry starts at 2). It's easier to raise matrices to powers after we diagonalize them, so find an invertible P and diagonal D so that $A = PDP^{-1}$ (the numbers are a little gross, so don't be alarmed).

2 cont.

(c) Since $A^k = PD^kP^{-1}$, we can write out F_n as the first component of $PD^{n-2}P^{-1}\begin{bmatrix}F_2\\F_1\end{bmatrix} = PD^{n-2}P^{-1}\begin{bmatrix}1\\0\end{bmatrix}$ (if we wanted to be clever, we could write this as

$$F_n = \begin{bmatrix}1 & 0\end{bmatrix} PD^{n-2}P^{-1}\begin{bmatrix}1\\0\end{bmatrix},$$

as the row vector picks out the first component). Use this to write down a formula for F_n (don't worry about multiplying out powers of any terms involving square roots, just leave them as whatever they are)! Nifty!!!

Under the Hood: The right way to think about the matrices P and P^{-1} is as change-of-coordinates matrices to an *eigenbasis*—then the requirement for diagonalizability is that the eigenvectors of A form a basis for the space they're in. Essentially, what we're doing is choosing a clever basis so that A looks like a diagonal matrix in that basis.

Inner Products, Length, and Orthogonality

The Punch Line: We can compute a real number relating two vectors—or a vector to itself—that gives information on both length and angle.

Warm-Up What are the lengths of these vectors, as found geometrically (using things like the Pythagorean Theorem)?

(a) $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(e) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(f) $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

The Inner Product: If we think about a vector $\vec{v} \in \mathbb{R}^n$ as a $n \times 1$ matrix (a single column), then $\vec{v}^T$ is a $1 \times n$ matrix (a single row, sometimes called a row vector). Then we can multiply $\vec{v}^T$ against a vector (on the left) to get a 1×1 matrix, which we can consider a scalar. This is the idea behind the *inner product* in $\mathbb{R}^n$, also called the *dot product*: we take two vectors, $\vec{u}$ and $\vec{v}$, and define their inner product as $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$. This corresponds to multiplying together corresponding entries in the vectors, then adding all of the results to get a single number.

1 Find the inner product of the two given vectors:

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(e) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} x \\ y \end{bmatrix}$

(f) $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} -y \\ x \end{bmatrix}$

Length and Orthogonality: We observe that in $\mathbb{R}^2$, the quantity $\sqrt{\vec{v}\cdot\vec{v}}$ is the length of $\vec{v}$ as given by the Pythagorean Theorem. This motivates us to define the length of a vector in *any* $\mathbb{R}^n$ as $\left\|\vec{v}\right\| = \sqrt{\vec{v}\cdot\vec{v}}$ (encouraged that it also agrees with our idea of length in $\mathbb{R}^1$ and $\mathbb{R}^3$). Then the *distance* between $\vec{u}$ and $\vec{v}$ is $\left\|\vec{u}-\vec{v}\right\|$, the length of the vector between them.

We also observe that in $\mathbb{R}^2$, if $\vec{u}$ and $\vec{v}$ are perpendicular then $\vec{u}\cdot\vec{v} = 0$, and vice versa. To generalize this, we say $\vec{u}$ and $\vec{v}$ are *orthogonal* if $\vec{u}\cdot\vec{v} = 0$ (and indeed, this matches with perpendicularity in three dimensions as well).

2 What are the lengths of these vectors (computed with inner products)?

(a) $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(b) $\begin{bmatrix} 2 \\ -3 \\ 1 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(d) The vector of all 1s in $\mathbb{R}^n$ (this is something of a challenge problem)

3 What is the distance between these two vectors? Are they orthogonal?

(a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -5 \end{bmatrix}$

(d) Two (different) standard basis vectors in $\mathbb{R}^n$

Under the Hood: This idea of orthogonality can be used to find the collection of *all* vectors which are orthogonal to some given $\vec{u}$. These are the solutions to the equation $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = 0$. This is just finding the nullspace of the matrix $\vec{u}^T$, but now it has a nice geometric interpretation. The solution set is a subspace, known as the *orthogonal complement* of $\vec{u}$.

Orthogonal Sets

The Punch Line: With an inner product, we can find especially nice bases called orthonormal sets.

Warm-Up What are the inner products and lengths of the following pairs of vectors?

(a) $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

(f) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Orthogonal and Orthonormal Sets: If the inner product of every pair of vectors in a set $\{\vec{u}_1, \ldots, \vec{u}_m\}$ is zero, we call the set *orthogonal*. In this case, it's a linearly independent set, and so a basis for its span. If there are n vectors in the set, it is a basis for $\mathbb{R}^n$.

If in addition to begin orthogonal, every vector in the set is a *unit vector* (has length 1), we call the set *orthonormal*. Since an orthogonal set is a basis, there is a unique representation of any vector $\vec{v} = c_1\vec{u}_1 + \cdots + c_n\vec{v}_n$; as it turns out the coefficients $c_i = \frac{\vec{u}_i \cdot \vec{v}}{\vec{u}_i \cdot \vec{u}_i}$. If the set is orthonormal, this means the coefficients are just the inner products with the basis vectors.

1 Are these sets orthogonal? If so, find an orthonormal set by rescaling them.

(a) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \\ -12 \end{bmatrix}, \begin{bmatrix} 3 \\ -11 \\ -7 \\ -6 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \\ \frac{1}{13} \end{bmatrix}, \begin{bmatrix} \sqrt{5} \\ 1 \\ 0 \\ 0 \\ -\sqrt{7} \end{bmatrix}, \begin{bmatrix} 83 \\ 18 \\ 27 \\ -1 \\ 0 \end{bmatrix} \right\}$

Orthogonal Matrices: In an unfortunate twist of terminology, we call a matrix an *orthogonal matrix* if its columns are an orthonormal set (not just orthogonal like the name might make you think). These matrices are precisely those matrices U where $U^T U = I_n$.

2 Are these matrices orthogonal?

(a) $\begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

(c) $\frac{1}{\sqrt{6}}\begin{bmatrix} 1 & 2 \\ -2 & -1 \\ 1 & -1 \end{bmatrix}$

(d) $\frac{1}{\sqrt{6}}\begin{bmatrix} 1 & -2 & 1 \\ 2 & -1 & -1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(f) The change-of-coordinates matrices to and from an orthonormal set [Challenge problem]

Under the Hood: Orthogonal transformations from $\mathbb{R}^n$ to itself are precisely those which do not change inner products (where $(U\vec{u}) \cdot (U\vec{v}) = \vec{u} \cdot \vec{v}$ for all pairs of vectors). This means they do not change the geometry involved (lengths, relative angles, or distances), so they are particularly interesting transformations.

This is an example of an incredibly common pattern in mathematics: when there is some kind of structure (like a vector space structure, or geometric relationships), mathematicians are interested in finding the collection of functions which preserve that structure (linear transformations and transformations by orthogonal matrices, in those two cases). There are also other classes of linear transformations that preserve things like areas (determinant has absolute value 1), or orientation (determinant is precisely 1), or just angles and not lengths (columns are orthogonal but not necessarily orthonormal), and many more.

Orthogonal Projection

The Punch Line: Inner products make it quite easy to compute the component of vectors that lie in interesting subspaces—in particular, components in the direction of any other vector.

Warm-Up What is the closest vector on the x-axis to the following vectors?

(a) $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ 9 \end{bmatrix}$ (c) $\begin{bmatrix} x \\ y \end{bmatrix}$

What is the closest point on the y-axis to these vectors? On the xy-plane?

(d) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ (e) $\begin{bmatrix} 9 \\ 1 \\ 2 \end{bmatrix}$ (f) $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Orthogonal Projection: If we have some vector $\vec{u}$ that we're interested in, we can compute the *orthogonal projection* of any other vector $\vec{v}$ onto $\vec{u}$ as $\frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$. That is, the ratio of the inner product of $\vec{v}$ and $\vec{u}$ to the inner product of $\vec{u}$ with itself is the coefficient on $\vec{u}$ giving the closest vector in $\text{Span}\{\vec{u}\}$ to $\vec{v}$. This coefficient can be thought of as "the amount of $\vec{v}$ in the direction of $\vec{u}$", and the projection (which is a vector) as "the component of $\vec{v}$ in the direction of $\vec{u}$."

1 Compute the projection of $\vec{v}$ onto $\vec{u}$.

(a) $\vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ onto $\vec{u} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

(b) $\vec{v} = \begin{bmatrix}2\\-1\end{bmatrix}$ onto $\vec{u} = \begin{bmatrix}1\\3\end{bmatrix}$

(c) $\vec{v} = \begin{bmatrix}2\\-1\end{bmatrix}$ onto $\vec{u} = \begin{bmatrix}1\\2\end{bmatrix}$

(d) $\vec{v} = \begin{bmatrix}1\\0\end{bmatrix}$ onto $\vec{u} = \begin{bmatrix}x\\y\end{bmatrix}$

Projection Onto Subspaces: If W is a subspace of $\mathbb{R}^n$, we can compute the projection of a vector onto W. This is found by taking all and only the component of a vector which lie in W, which is most easily done if we have an orthogonal (or orthonormal) basis for W. Then we can simply compute the relevant inner products to project onto each basis vector, then add up all the results. (Note that this won't work if the basis isn't orthogonal.)

2 Project the vector $\vec{v}$ onto the subspace spanned by the given vectors.

(a) $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

(b) $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Under the Hood: Why are orthogonal bases so much easier to project onto (we don't even have a good way to project onto the span of non-orthogonal vectors other than finding an orthogonal basis for that same subspace)? Heuristically, each vector in an orthogonal set is giving "independent information" about a vector in their span. Travelling in the direction of one of them doesn't move at all in the direction of the others, while for non-orthogonal vectors, increasing in one direction also moves in some of the others, and it's hard to separate the effects.

So, a basis gives enough information to describe any vector (it spans the space) and doesn't have redundant information (it's linearly independent), while an *orthogonal* basis also has the property that pieces of that description don't interfere with each other. An orthonormal basis is even nicer, in that the information requires less processing to get information about lengths—the coefficient on each component is the length in that direction (in other bases, the length of the basis vector changes this).

The Gram-Schmidt Process

The Punch Line: We can turn any basis into an orthonormal basis using a (relatively) simple procedure.

Warm-Up For what choices of the variables are these bases orthogonal? Can they be made orthonormal by choosing variables correctly?

(a) $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}x\\y\end{bmatrix}\right\}$

(b) $\left\{\begin{bmatrix}4\\y\end{bmatrix}, \begin{bmatrix}x\\1\end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix}1/2\\1/2\\1/2\end{bmatrix}, \begin{bmatrix}x\\y\\0\end{bmatrix}, \begin{bmatrix}-x\\0\\z\end{bmatrix}\right\}$

The Gram-Schmidt Process: Suppose we know $\{\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n\}$ is a basis for some subspace W we are interested in. We can make an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ for the same subspace by repeatedly stripping away the parts of vectors that are not orthogonal to the previous ones.

In particular, we set $\vec{v}_1 = \vec{w}_1$ (there aren't previous vectors that it could be nonorthogonal to). Then we set $\vec{v}_2 = \vec{w}_2 - \frac{\vec{w}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$ (we take off any part of $\vec{w}_2$ that's in the direction $\vec{v}_1$ with a projection). Similarly, we set $\vec{v}_3 = \vec{w}_3 - \frac{\vec{w}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{w}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$ (we have to remove parts in the first *two* directions now). In general, we set

$$\vec{v}_k = \vec{w}_k - \frac{\vec{w}_k \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \cdots - \frac{\vec{w}_k \cdot v_{k-1}}{\vec{v}_{k-1} \cdot v_{k-1}}\vec{v}_{k-1}$$

(subtracting off the projection onto all previous vectors in the basis we are constructing).

1 Apply the Gram-Schmidt Process to the following (ordered) bases:

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 17 \\ -3 \\ 1 \end{bmatrix} \right\}$

Orthonormal Bases: After applying the Gram-Schmidt Process, it's easy to get an orthonormal basis—just rescale the results. It's important to note that the rescaling can be done right after subtracting off the projections onto the previous vectors, but shouldn't be done before doing so, as subtracting vectors changes lengths (it won't harm the process, but you won't get unit vectors out of it).

2	Find the orthonormal bases from the results of Problem 1.