

Equivalent Systems

What is a system of linear equations? At first blush, it's something like this:

$$\begin{cases} x_1 + x_2 = 4, \\ x_1 - 2x_2 = 1. \end{cases} \quad (1)$$

We have a number of *variables* (here, x_1 and x_2), some *coefficients* multiplying those variables (here, 1 on x_1 and x_2 in the first equation, 1 on both x_1 and -2 on x_2 in the second), and some *constants* on their own (here, 4 and 1). Each variable is only present on its own, to the first power—there are no terms like x_1^2 , x_1x_2 , or $\sin(x_2)$ present.

Of course, we're not just interested in looking at the system: we want to solve it. It's worth thinking about what a "solution" is, even though you've probably got the right idea in mind. We're after values that we can plug in for the variables all at once to make every equation *simultaneously* true. That is, when we multiply those values by the appropriate coefficients and add them in the way described, we get as a result the same constants we were given.

The way I often think of this is to imagine that, on their own, each of the variables could take on any value (that is, could potentially be any real number), but each equation describes a *relationship* between the variables that *constrains* those values. Then finding a solution to an equation means finding values for the variables that satisfy the constraint represented by that relationship, and finding a solution to the system means finding values that satisfy *all* of the constraints.

What does this look like? For the system above there are only two variables, so we can picture the possible values of x_1 and x_2 as laid out on a plane. The x -coordinate gives the value for x_1 , and the y -coordinate gives the value for x_2 . Any particular pair of values for the two variables then corresponds to a point on that plane.

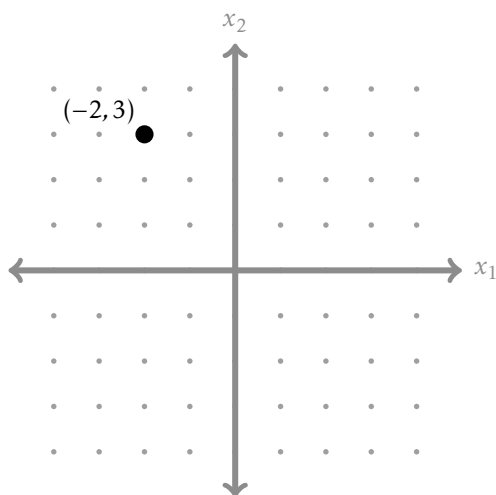


Figure 1: Representation of $x_1 = -2$, $x_2 = 3$.

The equation $x_1 + x_2 = 4$ gives the constraint that the values we can choose for x_1 and x_2 have to add up to four. This means that they have to be chosen from a line in the plane of possible values we could have chosen. Similarly, the equation $x_1 - 2x_2 = 1$ gives the constraint that the values for x_1 and x_2 have to be chosen from a (different) line.

There are infinitely many values that satisfy the first constraint separately from the second: anything on the line will do. It prescribes some relationship between the values we can choose for the two variables, but we still have some freedom. Similarly, there are infinitely many ways to satisfy the second constraint independently from the first. However, in order to satisfy both constraints simultaneously, we need to choose values for x_1 and x_2 that

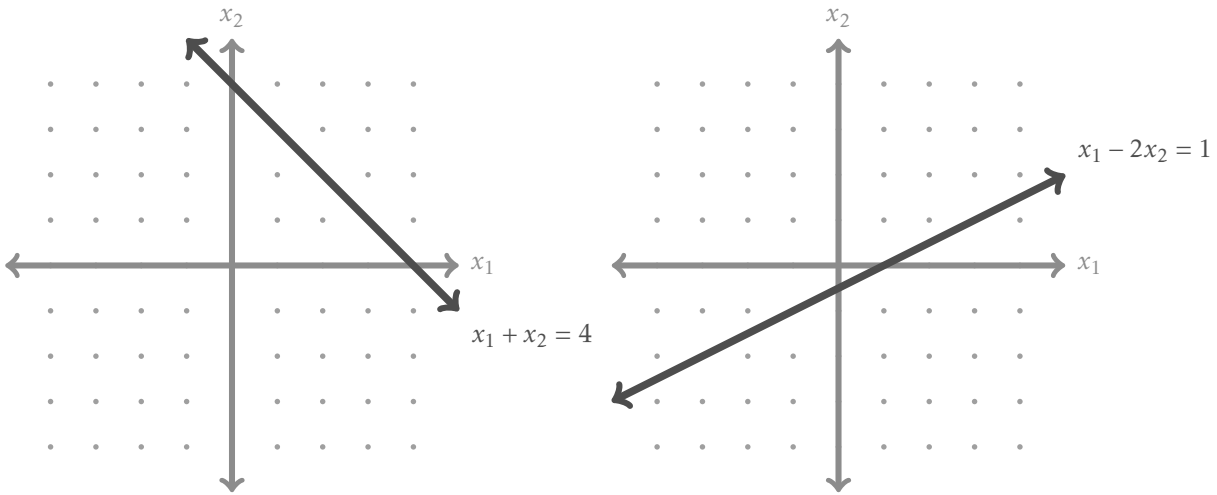


Figure 2: The two constraints of (1).

are on *both lines* simultaneously; that is, we need a point where the two lines intersect. There's only one point like this, $(3, 1)$, and it corresponds to setting $x_1 = 3$ and $x_2 = 1$.

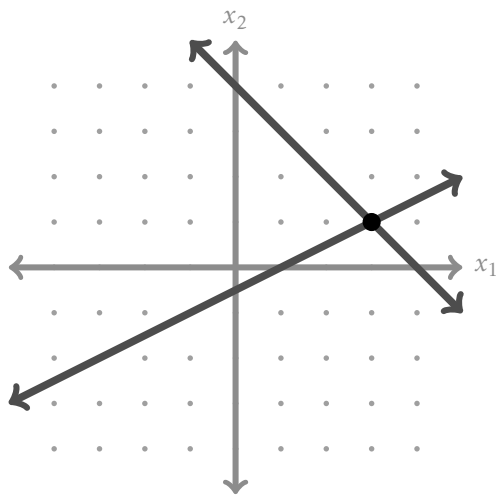


Figure 3: The solution of (1) at $(3, 1)$.

We've found for this system that the *solution set* (that is, all of the possible solutions, or settings of the variables satisfying all of the constraints) is pretty small: just a single point. We can think of the solution set as a feature of the system that we were given. Usually, though, the solution set is the information we actually care about, and the system is just a way of describing it.

If we take this viewpoint, then the system we were given is just one way of describing the information we're really after (the solution set). Quite possibly, there are many ways of describing the same set: if we solve a different system of equations and find the same solution set, we've found an *equivalent system*. Such a system is an equally valid description of the information we actually care about. For example, the following systems are both equivalent to the one we started with:

$$\begin{cases} x_1 - x_2 = 2, \\ 2x_1 + 5x_2 = 11, \end{cases} \quad (2)$$

$$\begin{cases} x_1 = 3, \\ x_2 = 1. \end{cases} \quad (3)$$

If we picture these in the plane, we get different pairs of lines representing the different equations, but the point of intersection remains the same. This is the graphical depiction of what it means for systems to be equivalent: the constraints (here lines) may be different individually, but the region where they are all satisfied (here the intersection point) remains the same.

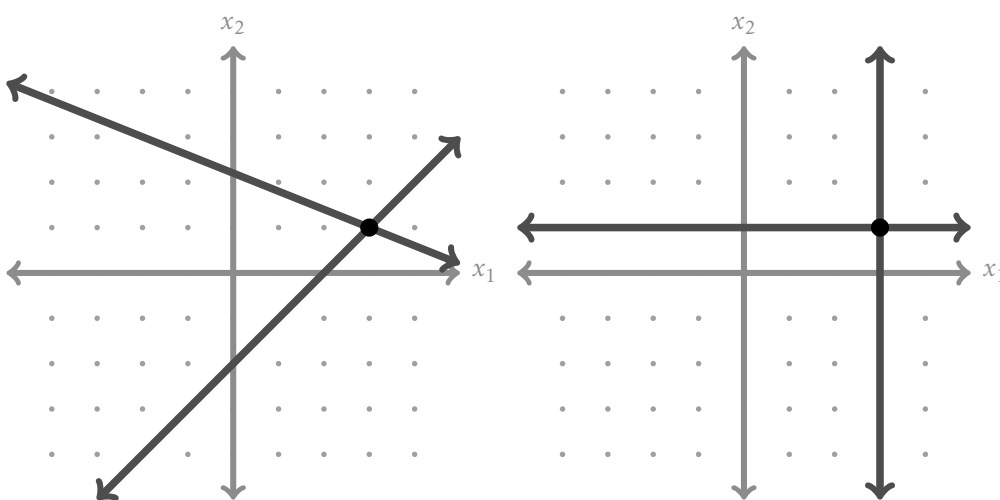


Figure 4: The solutions of (2) and (3).

Our quest is to find the “best description” of the solution set. In system (3), we don't have to do any work to determine what the point is, the system (because technically it is a system of linear equations) is just each coordinate listed in order. If the solution set is a single point, this is the ideal description we're after.

This kind of situation might arise when the easiest way to represent the information as it's given to us isn't the easiest way to represent what we want to know. For a (slightly contrived) example, perhaps we read in the newspaper that in a recent soccer game, two players from our favorite team scored:

“Team captain E. Noether¹ (jersey #1) and forward A. Lovelace² (jersey #2) scored a combined 4 goals in last night's game. In a stunning performance, Noether scored twice as many as Lovelace, and then put in another one to cap off the night!”

With the information given, it's natural to represent the information using system (1), putting x_1 and x_2 as the number of goals scored, even though we're probably not particularly invested in the particulars of how the newspaper described the game (the particular system chosen), and instead might just want to know how many goals each player scored (the solution set).

¹https://en.wikipedia.org/wiki/Emmy_Noether

²https://en.wikipedia.org/wiki/Ada_Lovelace

Sometimes, of course, a system of linear equations may be somewhat more complicated. It can have no solutions, like

$$\begin{cases} x_1 = 3, \\ x_2 = 3, \\ x_1 + x_2 = 1, \end{cases} \quad (4)$$

or have infinitely many solutions, like

$$\begin{cases} x_1 + x_2 = 0. \end{cases} \quad (5)$$

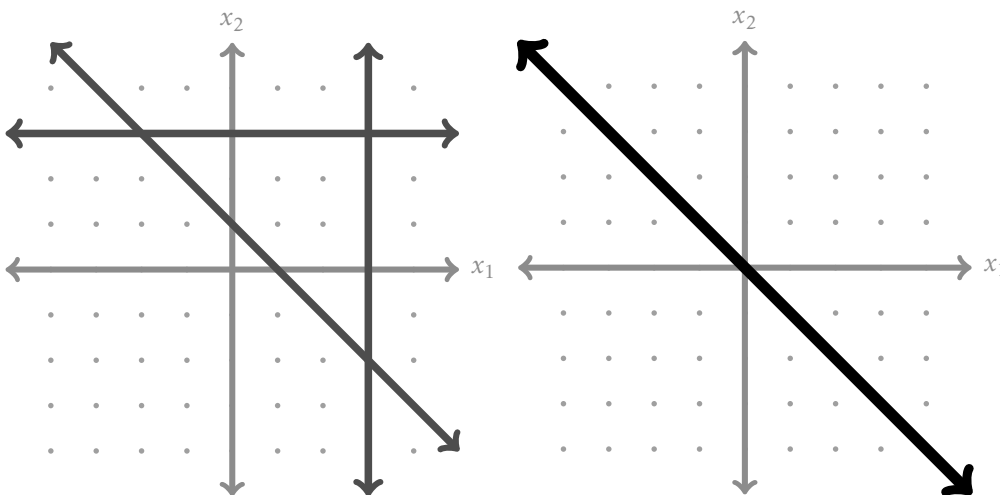


Figure 5: The constraints of (4) and solutions of (5).

Also, if a system has three variables, it becomes much more difficult to visualize (each constraint is satisfied by all values in a plane in 3D space for such a system), and it is all but impossible to visualize a system with four or more variables all at once.

The same ideas roughly hold, though: each equation gives a constraint that's line-like (this is why it's called *linear* algebra), and the solution set is where all of the constraints intersect. There can be many different systems (collections of constraints) that have the same solution sets (intersect in the same place), or a system can fail to have any solutions at all (if the constraints don't all intersect at once). All of this means that this kind of 2D model can be a useful starting point in thinking about what's going on with linear systems, and what it means for two linear systems to be equivalent.