5c Consider the orthogonal complement to a vector subspace W in $\mathbb{R}^3$. Find the orthogonal complement $W^\perp$ when $W = \{\vec{v} \mid \vec{w}\cdot\vec{v} = 0, \text{ with } \vec{w} = (5,3,0)\}$.

We're after the set $W^\perp$, which is in general $\{\vec{u} \mid \vec{u}\cdot\vec{v} = 0, \text{ with } \vec{v} \in W\}$. That is, all vectors $\vec{u}$ which are orthogonal to *every* vector $\vec{v}$ in W.

Method 1: The subspace W is defined in a very similar way, as all vectors which are orthogonal to $\vec{w} = (5,3,0)$. So, we're after everything orthogonal to everything orthogonal to $\vec{w}$. This means that $\vec{w}$ is certainly in $W^\perp$: it's orthogonal to everything orthogonal to it! In fact, $c\vec{w}$ for any $c \in \mathbb{R}$ is in $W^\perp$, geometrically because orthogonality is a statement about angles, and scaling length doesn't change angles, and algebraically because

$$\vec{v}\cdot(c\vec{w}) = \vec{v}^T(c\vec{w}) = c\vec{v}^T\vec{w} = c(\vec{v}\cdot\vec{w}) = 0$$

(because $\vec{v}^T$ is just a matrix, and we're assuming $\vec{v} \in W$, so it's orthogonal to $\vec{w}$). So, $\operatorname{span}\{\vec{w}\}$ is in $W^\perp$, and we just have to check if anything else is.

If $W^\perp$ had vectors in it that weren't in $\operatorname{span}(\vec{w})$, it would be at least two-dimensional, so we would be able to choose some orthogonal basis for $W^\perp$ involving $\vec{w}$ and some other vector $\vec{w}'$. But then $\vec{w}'\cdot\vec{w} = 0$ (because it's an orthogonal basis) so $\vec{w}' \in W$, so $\vec{w}'$ couldn't have been in $W^\perp$ after all! So, $W^\perp$ has to be just $\operatorname{span}\{\vec{w}\}$.

Method 2: We can figure out the general form for vectors in W: if we start by writing $\vec{v} = (v_1, v_2, v_3)$, we know we have to satisfy the equation $\vec{v}\cdot\vec{w} = 0$, so $(v_1,v_2,v_3)\cdot(5,3,0) = 5v_1 + 3v_2 = 0$, so $v_1 = -\frac{3}{5}\vec{v}_2$. We don't have any conditions on v_3, so we can see that any vector $\vec{v} \in W$ has the form $(-\frac{3}{5}v_2, v_2, v_3)$ for some pair of real numbers v_2 and v_3. This means in turn that $W = \operatorname{span}\{(-3,5,0),(0,0,1)\}$ (setting $v_2 = 5$ and $v_3 = 0$, and $v_2 = 0$ and $v_3 = 1$, respectively).

So, to be in $W^\perp$, a vector $\vec{u}$ must satisfy $(-3,5,0)\cdot\vec{u} = 0$ and $(0,0,1)\cdot\vec{u} = 0$. Putting these two linear equations into one matrix equation gives

$$\begin{bmatrix} -3 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\vec{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So, $W^\perp$ is precisely the null space of the matrix above, which one can work out to be $\operatorname{span}\{(5,3,0)\} = \operatorname{span}\{\vec{w}\}$.