A renormalized Riesz potential and applications

Mihai Putinar

Abstract. The convolution in R^n with $|x|^{-n}$ is a very singular operator. Endowed with a proper normalization, and regarded as a limit of Riesz potentials, it is equal to Dirac's distribution δ. However, a different normalization turns the non-linear operator:

$$E_f = exp(\frac{-2}{|S^{n-1}|}\ |x|^{-n} * f),$$

into a remarkable transformation. Its long history (in one dimension) and some of its recent applications in higher dimensions make the subject of this exposition. A classical extremal problem studied by A. A. Markov is related to the operation E in one real variable. Later, the theory of the spectral shift of self-adjoint perturbations was also based on E. In the case of two real variables, the transform E has appeared in operator theory, as a determinantal-characteristic function of certain close to normal operators. The interpretation of E in complex coordinates reveals a rich structure, specific only to the plane setting. By exploiting an inverse spectral function problem for hyponormal operators, applications of this exponential transform to image reconstruction, potential theory and fluid mechanics have recently been discovered. In any number of dimensions, the transformation E, applied to characteristic functions of domains Ω, can be regarded as the geometric mean of the distance function to the boundary. This interpretation has a series of unexpected geometric and analytic consequences. For instance, for a convex algebraic Ω, it turns out that the operation E is instrumental in converting finite external data (such as field measurements or tomographic pictures) into an equation of the boundary.

§1. Introduction

Riesz potentials, that is convolution operators with fractional powers of the distance to a point in $\mathbf{R}^n$, are related to several inverse problems of mathematical physics and geometry. The Newtonian potential in $\mathbf{R}^n$, $n \geq$

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3, is the best example. Riesz himself used these operators for writing the solution of the Cauchy problem for the wave equation in closed from. The modern theory of the Radon transform is based, both theoretically and numerically, on Riesz potentials. A common feature of the latter example is a uniqueness property of Riesz potentials $I^\alpha(\mu)$ of compactly supported measures μ. Namely, except for a discrete set of orders α, the germ of $I^\alpha(\mu)(x)$ at a point $x = x_0$ external to the support of μ determines μ. As noted by Riesz, this determination passes through the sequence of power moments of the measure. The present article is devoted to some specific, constructive aspects of this reconstruction process.

An extremal moment problem of A.A. Markov, for volumes in $\mathbf{R}^n$ carrying a shade function, offers the ground of exactly converting finitely many moments into the original measure. Roughly speaking, only the semi-algebraic sets given by a single polynomial inequality qualify for this exact reconstruction. The theoretical basis for Markov's problem was put more than half a century ago by Akhiezer and Krein [5].

In dimension $n = 1$ the relevant phenomena were discovered by Markov. Nowadays, due to the modern language of function theory and functional analysis, Markov's results can be organized around three different representations (additive, multiplicative and Hilbert space theoretic) of the Cauchy transforms of positive measures of compact support on the real line (see Proposition 4.1). The best rational approximation of these Cauchy transforms, also known as the Padé approximation, can conveniently be interpreted in terms of their Hilbert space representation (see Section 5).

Fortunately, the next dimension $n = 2$ also has specific, non-trivial results. They are mainly due to the progress in the theory of hyponormal operators, and separately because of the rich theory of functions of a complex variable. The reconstruction algorithm of an area from its moments again involves a renormalization of the Riesz potential at the critical exponent $\alpha = 0$; the algorithm is exact on quadrature domains, a remarkable class of planar domains which has appeared at the crossroads of several branches of mathematics and physics (see Section 7). Some recent numerical experiments support this algorithm.

By considering a similar renormalized potential, and its exponential, in dimensions $n \geq 3$, a series of interesting geometric and analytic facts were recently discovered. They are briefly enumerated in Section 9. Altough much remains to be done, the global picture in $\mathbf{R}^n$, $n \geq 3$, is promising for applications.

The statements included in this article have at most a sketch of a proof. We have tried to give accurate references to all assertions and to guide the reader through the essential bibliography for the many independent theories alluded to.

§2. Riesz potentials

In this section we recall some basic properties of Riesz potentials, cf. [29], [36], [59]. Let $\alpha < n$ be a fixed parameter and denote:

$$I_\alpha(x) = \frac{\Gamma(\frac{n-\alpha}{2})}{2^\alpha \pi^{n/2}\Gamma(\frac{\alpha}{2})} \frac{1}{|x|^{n-\alpha}}.$$

This is a locally integrable function in $\mathbf{R}^n$, which can be regarded as a tempered distribution $I_\alpha \in \mathcal{S}'$. The apparently complicated coefficient was chosen so that the Fourier transform (with the correct normalization) is simply:

$$\hat{I_\alpha}(\xi) = \frac{1}{|\xi|^\alpha}.$$

The map $\alpha \mapsto I_\alpha \in \mathcal{S}'(\mathbf{R}^n)$ extends meromorphically to the whole complex plane, with poles only at $\alpha = n, n+2, n+4, \ldots$. The Fourier transform image of the functions I_α immediately reveals the identities:

$$-\Delta I_\alpha = I_{\alpha-2},$$

$$I_\alpha * I_\beta = I_{\alpha+\beta},$$

$$I_0 = \delta,$$

whenever the values $\alpha = n, n+2, \ldots$ do not occur.

Thus, I_α is a semigroup with respect to convolution, which, for $n > 2$, interpolates in an analytic scale Dirac's delta distribution and the fundamental solution I_2 for the Laplace operator Δ: $\Delta I_2 = -\delta$.

For a Radon measure μ of, say, compact support in $\mathbf{R}$, the potential

$$I_\alpha^\mu = I_\alpha * \mu, \quad \alpha < n,$$

is well defined. It is a continuous function on $\mathbf{R}^n$, even real analytic outside the support of μ. For $n > 2$ and $\alpha = 2$ we recover the usual Newtonian potential of the measure μ:

$$U^\mu(x) = I_2^\mu(x) = \frac{1}{(n-2)|S^{n-1}|} \int \frac{d\mu(y)}{|y-x|^{n-2}}.$$

The case $n = 1$ corresponds to the classical integrals

$$I_\alpha^\mu(x) = \text{const.} \int_{-\infty}^{\infty} |y-x|^{\alpha-1} d\mu(y),$$

already present in the works of Liouville and Riemann.

The two large articles of Marcel Riesz [48], [49], where he has introduced the functions I_α and the convolution operators (i.e. the potentials) defined by them, contain a detailed extension of the Newtonian potential theory to this new setting and applications of the same functions to solving in close form the Cauchy problem for the wave equation. Later on, both ideas have flourished, cf. [31], [36]. For classical analysis, the Riesz potentials I_α are regarded as fractional integration operators and they appear in the study of the degrees of smoothness of real functions, see [59].

For this article, two uniqueness principles discovered by Riesz are relevant, see Sections 10, 11 of Chapter III in [48].

Theorem (Riesz). *Let $n > 2$ and let μ be a Radon measure with compact support in $\mathbf{R}^n$. Assume that α is not congruent to 2 or n modulo 2. If $I^\mu_\alpha(x)$ is identically zero on a connected component of $\mathbf{R}^n \setminus \mathrm{supp}(\mu)$, then the measure μ is zero.*

The second uniqueness principle asserts essentially the same thing under the assumption that the germ at infinity of $I^\mu_\alpha(x)$ vanishes. Both proofs rely on the observation that repeated differentiation of $I^\mu_\alpha(x)$ and evaluation at a fixed point of $\mathbf{R}^n \setminus \mathrm{supp}(\mu)$ determines the moments:

$$a_\sigma(\mu) = \int x^\sigma d\mu(x), \quad \sigma \in \mathbf{N}^n.$$

Above we have adopted the multi-index notation: $x^\sigma = x_1^{\sigma_1}...x_n^{\sigma_n}, |\sigma| = \sigma_1 + \sigma_2 + \ldots + \sigma_n,\ \sigma_k \in \mathbf{N}$. The orders $\alpha = 2, 4, \ldots$ have to be excluded, because of the known non-uniqueness phenomenon for Newtonian potentials:

$$U^\mu(x) = U^\nu(x), \quad |x| > R,$$

does not imply in general that $\mu = \nu$. For instance μ can be a point mass at $x = 0$ and ν a uniformly distributed mass on a ball centered at $x = 0$, yet they are not distinguished by their external Newtonian potentials (a fact already known to Newton).

The main theme of this survey is to show how Riesz potentials can be instrumental in reconstructing a (positive, compactly supported) measure μ from its moments

$$a(\mu) = (a_\sigma(\mu))_{\sigma \in \mathbf{N}^n}.$$

This is in itself a fundamental problem, whose importance was well recognized by both mathematicians and their more applied customers.

To illustrate the ubiquity of the Riesz potentials in reconstruction problems we reproduce (e.g. from Chapter II of [38]) the well-known inversion formula for the Radon transform, see also [28], [29]. If one defines the Radon transform of a function f by:

$$(Rf)(\omega, s) = \int_{\langle x,\omega\rangle = s} f(x)dx, \quad f \in \mathcal{S}(\mathbf{R}^n),$$

and its adjoint by

$$(R^\sharp_\omega g)(x) = g(\langle x, \omega\rangle), \quad g \in \mathcal{S}(\mathbf{R}), \|\omega\| = 1,$$

then, for $\alpha < n$,

$$f = cI_{-\alpha} * R^\sharp I_{\alpha-n+1} * 5Rf,$$

where c is a universal constant and $*$ denotes convolution.

Let μ be a Radon measure with compact support in $\mathbf{R}^n$. The object of interest throughout this article is the transformation:

$$E_\mu(x) = \exp(-\frac{2}{|S^{n-1}|}\int \frac{d\mu(y)}{|y-x|^n}), \quad x \in \operatorname{supp}(\mu)^c. \qquad (2.1)$$

As an application of the proof of Riesz' uniqueness theorem, the germ at infinity of the real analytic function E_μ determines μ.

The most interesting example for us will be a uniformly distributed mass on a bounded domain $\Omega \subset \mathbf{R}^n$. In this case we simply write:

$$E_\Omega(x) = \exp(-\frac{2}{|S^{n-1}|}\int_\Omega \frac{dy}{|y-x|^n}), \quad x \in \Omega^c. \qquad (2.2)$$

We will see later that, although the integral (2.2) produces a logarithmic singularity in x when this variable tends from outside to a smooth portion of the boundary $\partial\Omega$, the exponential restores the smoothness in x, even up to real analyticity.

This article is devoted to some constructive aspects of the determination $E_\mu \mapsto \mu$.

§3. The abstract L problem of moments

The classical L-problem of moments offers a good theoretical motivation for applying the exponential transform (2.1) to reconstructing extremal measures μ from their moments, or equivalently, from the germ at infinity of E_μ. The material below is classical and can be found in the monographs [5], [32], [35]. We present only a simplified version of the abstract L-problem.

Let K be a compact subset of $\mathbf{R}^n$ with interior points and let $A \subset \mathbf{N}^n$ be a finite subset of multi-indices. We are interested in the set Σ_A of moment sequences $a(f) = (a_\sigma(f))_{\sigma\in A}$:

$$a_\sigma(f) = \int_K x^\sigma f(x)dx, \quad \sigma \in A,$$

of all measurable functions $f : K \longrightarrow [0,1]$. Regarded as a subset of $\mathbf{R}^{|A|}$, Σ_A is a compact convex set. An $L^1 - L^\infty$ duality argument (known as the abstract L-problem of moments) shows that every extremal point of Σ_A is a characteristic function of the form:

$$\chi_{\{p<\gamma\}},$$

where we denote:

$$\{p < \gamma\} = \{x \in K;\ p(x) < \gamma\}.$$

Above γ is a real constant and p is an A-polynomial with real coefficients, that is $p(x) = \sum_{\sigma \in A} c_\sigma x^\sigma$. For proofs the reader can consult [5] pp. 175-204, or [32], or [35].

As a consequence, the above description of the extremal points in the moment set Σ_A implies the following remarkable uniqueness theorem.

Theorem (Akhiezer and Krein). *For each characteristic function χ of a level set in K of an A-polynomial there exists exactly one class of functions f in $L^\infty(K)$ satisfying $a(f) = a(\chi)$. For a non-extremal point $a(f) \in \Sigma_A$ there are infinitely many non-equivalent classes in $L^\infty(K)$ having the same A-moments.*

Let us consider a simple example:

$$K = \{(x, y);\ x^2 + y^2 \leq 1\} \subset \mathbf{R}^2,$$

and

$$\Omega_+ = \{(x, y) \in K;\ x > 0,\ y > 0\},\quad \Omega_- = \{(x, y) \in K;\ x < 0,\ y < 0\}.$$

The reader can prove by elementary means that the sets $\Omega_\pm$ cannot be defined in the unit ball K by a single polynomial inequality. On the other hand, the set

$$\Omega = \Omega_+ \cup \Omega_- = \{(x, y);\ xy > 0\},$$

is defined by a single equation of degree two.

Thus, no matter how the finite set of indices $A \subset \mathbf{N}^2$ is chosen, there is a continuum f_s, $s \in \mathbf{R}$, of essentially distinct measurable functions $f_s : K \longrightarrow [0,1]$ possessing the same A-moments:

$$\int_K x^{\sigma_1} y^{\sigma_2} f_s(x, y)\,dxdy = \int_{\Omega_+} x^{\sigma_1} y^{\sigma_2}\,dxdy,\quad s \in \mathbf{R},\ \sigma \in A.$$

On the contrary, if the set of indices A contains $(0,0)$ and $(1,1)$, then for every measurable function $f : K \longrightarrow [0,1]$ satisfying

$$\int_K x^{\sigma_1} y^{\sigma_2} f(x, y)\,dxdy = \int_\Omega x^{\sigma_1} y^{\sigma_2}\,dxdy,\quad \sigma \in A,$$

we infer by Akhiezer and Krein's Theorem that $f = \chi_\Omega$, almost everywhere.

On a more theoretical side, we can interpret Akhiezer and Krein's Theorem in terms of geometric tomography, see [18]. Fix a unit vector $\omega \in \mathbf{R}^n$, $\|\omega\| = 1$, and let us consider the parallel Radon transform of a function $f : K \longrightarrow [0, 1]$, along the direction ω:

$$(Rf)(\omega, s) = \int_{\langle x,\omega\rangle = s} f(x)dx.$$

Accordingly, the k-th moment in the variable s of the Radon transform is, for a sufficiently large constant M:

$$\int_{-M}^{M} (Rf)(\omega, s)s^k ds = \int_K \langle x, \omega\rangle^k f(x)dx =$$

$$\sum_{|\sigma|=k} \frac{|\sigma|!}{\sigma!} \int_K x^\sigma \omega^\sigma f(x)dx = \sum_{|\sigma|=k} \frac{|\sigma|!}{\sigma!} \omega^\sigma a_\sigma(f). \tag{3.1}$$

Since there are $N(n, d) = C^n_{n+d}$ linearly independent polynomials in n variables of degree less than or equal to d, a Vandermonde determinant argument shows, via the above formula, that the same number of different parallel projections of the "shade" function $f : K \longrightarrow [0, 1]$, determine, via a matrix inversion, all moments:

$$a_\sigma(f), \quad |\sigma| \leq d.$$

And of course, the converse holds, by formula (3.1). These transformations are known and currently used in image processing, see for instance [19] and [27] and the references cited there.

In conclusion, Akhiezer and Krein's Theorem asserts then that in the measurement process

$$f \mapsto ((Rf)(\omega_j, s))_{j=1}^{N(n,d)} \mapsto (a_\sigma(f))_{|\sigma|\leq d}$$

only black and white pictures, delimited by a single algebraic equation of degree less than or equal to d, can be exactly reconstructed. Even when these uniqueness conditions are met, the details of the reconstruction from moments are delicate. We shall see some examples in the next sections.

§4. Markov's extremal problem and the phase shift

By going back to the source and dropping a few levels of generality, we recall Markov's original moment problem and some of its modern interpretations. Again, this material is well exposed in the monographs [5] and [35].

Let us consider, for a fixed positive integer n, the L-moment problem on the line:

$$a_k = a_k(f) = \int_{\mathbf{R}} t^k f(t)dt, \quad 0 \le k \le 2n,$$

where the unknown function f is measurable, admits all moments up to degree $2n$ and satisfies:

$$0 \le f \le L, \text{ a.e..}$$

As noted by Markov, the next formal series transform is necessary in solving this question:

$$\exp[\frac{1}{L}(\frac{a_0}{z} + \frac{a_1}{z^2} + \dots \frac{a_{2n}}{z^{2n+1}})] = 1 + \frac{b_0}{z} + \frac{b_1}{z^2} + \dots . \tag{4.1}$$

Remark that, although the series under the exponential is finite, the resulting one might be infinite.

The following theorem is classical, see for instance [5] pp. 77-82. Its present form was refined by Akhiezer and Krein; partial similar attempts are due, among others, to Boas, Ghizzetti, Hausdorff, Kantorovich, Verblunsky and Widder, see [5], [35].

Theorem (Markov). *Let $a_0, a_1, \dots, a_{2n}$ be a sequence of real numbers and let $b_0, b_1, \dots$ be its exponential L-transform. Then there is an integrable function f, $0 \le f \le L$, possessing the moments $a_k(f) = a_k$, $0 \le k \le 2n$, if and only if the Hankel matrix $(b_{k+l})_{k,l=0}^n$ is non-negative definite. Moreover, the solution f is unique if and only if* $\det(b_{k+l})_{k,l=0}^n = 0$. *In this case the function f/L is the characteristic function of a union of at most n bounded intervals.*

The reader will recognize above a concrete validation of the abstract moment problem discussed in the previous section.

In order to better understand the nature of the L-problem, we interpret below the exponential transform from two different and complementary points of view. For simplicity we take the constant L to be equal to 1 and consider only compactly supported originals f, due to the fact that the extremal solutions have anyway compact support. Let μ be a positive Borel measure on $\mathbf{R}$, with compact support. Its Cauchy transform

$$F(z) = 1 - \int_{\mathbf{R}} \frac{d\mu(t)}{t - z},$$

provides an analytic function on $\mathbf{C} \setminus \mathbf{R}$ which is also regular at infinity, and has the normalizing value 1 there. The power expansion, for large values of $|z|$, yields the generating moment series of the measure μ:

$$F(z) = 1 + \frac{b_0(\mu)}{z} + \frac{b_1(\mu)}{z^2} + \frac{b_2(\mu)}{z^3} + \ldots.$$

On the other hand,

$$\mathrm{Im}F(z) = -\mathrm{Im}z \int \frac{d\mu(t)}{|t-z|^2},$$

whence

$$\mathrm{Im}F(z)\ \mathrm{Im}z < 0, \quad z \in \mathbf{C} \setminus \mathbf{R}.$$

Thus the main branch of the logarithm $\log F(z)$ exists in the upper half-plane and its imaginary part, equal to the argument of $F(z)$, is bounded from below by $-\pi$ and from above by 0. According to Fatou's theorem, the non-tangential boundary limits

$$f(t) = \lim_{\epsilon \to 0} \frac{-1}{\pi} \mathrm{Im}\ \log F(t + i\epsilon),$$

exist and produce a measurable function with values in the interval $[0, 1]$. According to Riesz-Herglotz formula for the upper-half plane, we obtain:

$$\log F(z) = -\int_{\mathbf{R}} \frac{f(t)dt}{t-z}, \quad z \in \mathbf{C} \setminus \mathbf{R}.$$

Or equivalently,

$$F(z) = \exp\left(-\int_{\mathbf{R}} \frac{f(t)dt}{t-z}\right).$$

One step further, let us consider the Lebesgue space $L^2(\mu)$ and the bounded self-adjoint operator $A = M_t$ of multiplication by the real variable. The vector $\xi = \mathbf{1}$ corresponding to the constant function 1 is A-cyclic, and according to the spectral theorem:

$$\int_{\mathbf{R}} \frac{d\mu(t)}{t-z} = \langle (A-z)^{-1}\xi, \xi \rangle, \quad z \in \mathbf{C} \setminus \mathbf{R}.$$

As a matter of fact an arbitrary function F which is analytic on the Riemann sphere minus a compact real segment, and which maps the upper/lower half-plane into the opposite half-plane has one of the above forms. These functions are known in rational approximation theory as *Markov functions.* See for instance [6].

In short, putting together the above comments we can state the following result.

Proposition 4.1. *The canonical representations:*

$$F(z) = 1 - \int_{\mathbf{R}} \frac{d\mu(t)}{t-z} =$$

$$\exp(-\int_{\mathbf{R}} \frac{f(t)dt}{t-z}) =$$

$$1 - \langle (A-z)^{-1}\xi, \xi \rangle$$

establish constructive equivalences between the following classes:

a). Markov's functions F(z);
b). Positive Borel measures μ of compact support on $\mathbf{R}$;
c). Functions $f \in L^{\infty}_{\mathrm{comp}}(\mathbf{R})$ of compact support, $0 \leq f \leq 1$;
d). Pairs (A, ξ) of bounded self-adjoint operators with a cyclic vector ξ.

The extremal solutions correspond, in each case exactly, to:

a). Rational Markov functions F;
b). Finitely many point masses μ;
c). Characteristic functions f of finitely many intervals;
d). Pairs (A, ξ) acting on a finite dimensional Hilbert space.

For a complete proof see for instance Chapter VIII of [37] and the references cited there.

The above dictionary is remarkable in many ways. Each of its terms has intrinsic values. They were long ago recognized in moment problems, rational approximation theory or perturbation theory of self-adjoint operators.

For instance, when studying the change of the spectrum under a rank-one perturbation $A \mapsto B = A - \xi\langle \cdot, \xi \rangle$ one encounters the *perturbation determinant*:

$$\Delta_{A,B}(z) = \det[(A - \xi\langle \cdot, \xi \rangle - z)(A-z)^{-1}] = 1 - \langle (A-z)^{-1}\xi, \xi \rangle.$$

The above exponential representation leads to the *phase-shift* function $f_{A,B}(t) = f(t)$:

$$\Delta_{A,B}(z) = \exp(-\int_{\mathbf{R}} \frac{f_{A,B}(t)dt}{t-z}).$$

The phase shift of, in general, a trace-class perturbation of a self-adjoint operator has certain invariance properties; it reflects by fine qualitative properties the nature of change in the spectrum. The theory of perturbation determinants and of the phase shift is nowadays well developed, mainly for its applications to quantum physics, see [34] and [57].

The reader will recognize above an analytic continuation in the complex plane of the real exponential transform

$$F(x) = E_f(x) = \exp(-\int_{\mathbf{R}} \frac{f(t)dt}{|t-x|}),$$

assuming for instance that $x < M$ and the function f is supported by $[M, \infty)$.

To give the simplest, yet essential, example, we consider a positive number r and the various representations of the function:

$$F(z) = 1 + \frac{r}{z} = \frac{z+r}{z} =$$

$$1 - \int_{\mathbf{R}} \frac{r d\delta_0(t)}{t-z} =$$

$$\exp[-\int_{-r}^{0} \frac{dt}{t-z}] =$$

$$\det[(-r-z)(-z)^{-1}].$$

In this case the underlying Hilbert space has dimension one and the two self-adjoint operators are $A = 0$ and $A - \xi\langle\cdot,\xi\rangle = -r$.

§5. The reconstruction algorithm in one real variable

Returning to our main theme, and as a direct continuation of the previous section, we are interested in the exact reconstruction of the original $f : \mathbf{R} \longrightarrow [0,1]$ from a finite set of its moments, or equivalently, from a Taylor polynomial of E_f at infinity. The algorithm described in this section is the diagonal Padé approximation of the exponential transform of the moment sequence. Its convergence, even beyond the real axis, is assured by a classical theorem of Markov.

Let $a_0, a_1, \ldots, a_{2n}$ be a sequence of real numbers with the property that its exponential transform:

$$\exp[\frac{1}{L}(\frac{a_0}{z} + \frac{a_1}{z^2} + \ldots \frac{a_{2n}}{z^{2n+1}})] = 1 + \frac{b_0}{z} + \frac{b_1}{z^2} + \ldots,$$

produces a non-negative Hankel matrix $(b_{k+l})_{k,l=0}^{n}$.

According to Markov's Theorem (see Section 4), there exists at least one bounded self-adjoint operator $A \in L(H)$, with a cyclic vector ξ, such that:

$$\exp[\frac{1}{L}(\frac{a_0}{z} + \frac{a_1}{z^2} + \ldots \frac{a_{2n}}{z^{2n+1}})] =$$

$$1 + \frac{\langle \xi, \xi \rangle}{z} + \frac{\langle A\xi, \xi \rangle}{z^2} + \dots \frac{\langle A^{2n}\xi, \xi \rangle}{z^{2n+1}} + O(\frac{1}{z^{2n+2}}).$$

Let $k < n$ and H_k be the Hilbert subspace spanned by the vectors $\xi, A\xi, \dots, A^{k-1}\xi$. Suppose that $\dim H_k = k$, which is equivalent to saying that $\det(b_{i+j})_{i,j=0}^{k-1} \neq 0$. Let π_k be the orthogonal projection of H onto H_k and let $A_k = \pi_k A \pi_k$. Then

$$\langle A_k^{i+j}\xi, \xi \rangle = \langle A_k^i \xi, A_k^j \xi \rangle =$$

$$\langle A^i \xi, A^j \xi \rangle = \langle A^{i+j}\xi, \xi \rangle,$$

whenever $0 \leq i, j \leq k-1$. In other terms, for large values of $|z|$:

$$\langle (A - z)^{-1}\xi, \xi \rangle = \langle (A_k - z)^{-1}\xi, \xi \rangle + O(\frac{1}{z^{2k+1}}). \tag{5.1}$$

By construction, the vector ξ remains cyclic for the matrix $A_k \in L(H_k)$. Let $q_k(z)$ be the minimal polynomial of A_k, that is the monic polynomial of degree k which annihilates A_k. In particular,

$$q_k(z)\langle (A_k - z)^{-1}\xi, \xi \rangle = \langle (q_k(z) - q_k(A_k))(A_k - z)^{-1}\xi, \xi \rangle = p_{k-1}(z)$$

is a polynomial of degree $k - 1$.

The two observations yield:

$$q_k(z)\langle (A - z)^{-1}\xi, \xi \rangle =$$

$$q_k(z)\langle (A_k - z)^{-1}\xi, \xi \rangle + O(\frac{1}{z^{k+1}}) =$$

$$p_{k-1}(z) + O(\frac{1}{z^{k+1}}).$$

The resulting rational function $R_k(z) = \frac{p_{k-1}(z)}{q_k(z)}$ is characterized by the property:

$$1 + \frac{b_0}{z} + \frac{b_1}{z^2} + \dots = 1 + R_k(z) + O(\frac{1}{z^{2k+1}}); \tag{5.2}$$

it is known as the *Padé approximation* of order $(k-1, k)$, of the given series.

A basic observation is now in order: since $b_0, b_1, \dots, b_{2k+1}$ is the power moment sequence of a positive measure, q_k is the associated orthogonal polynomial of degree k and p_k is a second order orthogonal polynomial of degree $k-1$. In particular their roots are simple and interlaced. We prove only the first assertion, the second one being of a similar nature, see for instance [4]. Indeed, let μ be the spectral measure of A localized at the vector ξ, exactly as in Proposition 4.1. Then, for $j < k$,

$$\int_{\mathbf{R}} t^j q_k(t)dt = \langle A^j \xi, q_k(A)\xi\rangle = \langle A_k^j \xi, q_k(A_k)\xi\rangle = 0.$$

Assume now that we are in the extremal case $\det(b_{i+j})_{i,j=0}^n = 0$ and that n is the smallest integer with this property, that is $\det(b_{i+j})_{i,j=0}^{n-1} \neq 0$. Since

$$b_{i+j} = \langle A^i \xi, A^j \xi\rangle,$$

this means that the vectors $\xi, A\xi, \ldots, A^n\xi$ are linearly dependent. Or equivalently that $H_n = H$ and consequently $A_n = A$.

According to the dictionary established by Proposition 4.1, this is another proof that the extremal case of the truncated moment 1-problem with data $a_0, a_1, \ldots, a_{2n}$ admits a single solution. The unique function $f : \mathbf{R} \longrightarrow [0,1]$ with this string of moments will then satisfy:

$$\exp(-\int_{\mathbf{R}} \frac{f(t)dt}{t-z}) = 1 + R_n(z) =$$

$$1 - \sum_{i=1}^n \frac{r_i}{a_i - z} =$$

$$\det[(A - \xi\langle\cdot,\xi\rangle - z)(A-z)^{-1}] =$$

$$\prod_{i=1}^n \frac{b_i - z}{a_i - z},$$

where the spectrum of the matrix A is $\{a_1, \ldots, a_n\}$, that of the perturbed matrix $B = A - \xi\langle\cdot,\xi\rangle$ is $b_1, \ldots, b_n$ and r_i are positive numbers. Again, one can easily prove that $b_1 < a_1 < b_2 < a_2 < \ldots < b_n < a_n$. By the last example considered in Section 4, we infer:

$$f = \sum_{i=1}^n \chi_{[b_i, a_i]},$$

or equivalently

$$f = \frac{1}{2}[1 - \mathrm{sign}\frac{p_{k-1} + q_k}{q_k}].$$

The above computations can therefore be put into a (robust) reconstruction algorithm of all extremal functions f. The Hilbert space method outlined above has other benefits, too. We illustrate them with a proof of another celebrated result due to A. A. Markov, and related to the convergence of the mentioned algorithm, in the case of non-extremal functions.

Theorem (Markov). *Let μ be a positive measure, compactly supported on the real line and let $F(z) = \int_{\mathbf{R}} (t-z)^{-1} d\mu(t)$ be its Cauchy transform. Then the diagonal Padé approximation $R_n(z) = p_{n-1}(z)/q_n(z)$ converges to $F(z)$ uniformly on compact subsets of $\mathbf{C} \setminus \mathbf{R}$.*

Proof: Let A be the multiplication operator with the real variable on the Lebesgue space $H = L^2(\mu)$ and let $\xi = \mathbf{1}$ be its cyclic vector. The subspace generated by $\xi, A\xi, ..., A^{n-1}\xi$ will be denoted as before by H_n and the corresponding compression of A by $A_n = \pi_n A \pi_n$.

If there exists an integer n such that $H = H_n$, then the discussion preceding the theorem shows that $F = R_n$ and we have nothing else to prove. Assume the contrary, that is the measure μ is not finite atomic.

Let $p(t)$ be a polynomial function, regarded as an element of H. Then

$$(A - A_n)p(t) = tp(t) - (\pi_n A \pi_n)p(t) = tp(t) - tp(t) = 0$$

provided that $\deg(p) < n$. Since $\|A_n\| \leq \|A\|$ for all n, and by Weierstrass Theorem, the polynomials are dense in H, we deduce:

$$\lim_{n\to\infty} \|(A - A_n)h\| = 0, \quad h \in H.$$

Fix a point $a \in \mathbf{C} \setminus \mathbf{R}$ and a vector $h \in H$. Then

$$\lim_{n\to\infty} \|[(A-a)^{-1} - (A_n - a)^{-1}]h\| =$$

$$\lim_{n\to\infty} \|(A_n - a)^{-1}(A - A_n)(A-a)^{-1}h\| \leq$$

$$\lim_{n\to\infty} \frac{1}{|\mathrm{Im}\ a|} \|(A - A_n)(A-a)^{-1}h\| = 0.$$

A repeated use of the same argument shows that, for every $k \geq 0$,

$$\lim_{n\to\infty} \|[(A-a)^{-k} - (A_n - a)^{-k}]h\| = 0.$$

Choose a radius $r < |\mathrm{Im}\ a| \leq \|(A_n - a)^{-1}\|^{-1}$, so that the Neumann series

$$(A_n - z)^{-1} = (A_n - a - (z-a))^{-1} = \sum_{k=0}^{\infty} (z-a)^k (A_n - a)^{-k-1}$$

converges uniformly and absolutely, in n and z, in the disk $|z - a| \leq r$. Consequently, for a fixed vector $h \in H$,

$$\lim_{n\to\infty} \|(A_n - z)^{-1}h - (A-z)^{-1}h\| = 0,$$

uniformly in z, $|z-a| \leq r$. In particular,

$$\lim_{n\to\infty} R_n(z) = \langle (A_n - z)^{-1}\xi, \xi\rangle = \lim_{n\to\infty} \langle (A_n - z)^{-1}\xi, \xi\rangle =$$

$$\langle (A-z)^{-1}\xi, \xi\rangle = F(z),$$

uniformly in z, $|z-a| \leq r$. $\square$

Details and a generalization of the above operator theory approach to Markov theorem can be found in [46]. See also Section 8 below.

§6. The exponential transform in two dimensions

The case of two real variables is special, partly due to the existence of a complex variable in $\mathbf{R}^2$. Let $g : \mathbf{C} \longrightarrow [0,1]$ be a measurable function and let $dA(\zeta)$ stand for the Lebesgue area measure. The exponential transform becomes, in complex variable notation:

$$E_g(z) = \exp(-\frac{1}{\pi}\int_{\mathbf{C}} \frac{g(\zeta)dA(\zeta)}{|\zeta - z|^2}), \quad z \in \mathbf{C} \setminus \mathrm{supp}(g).$$

This expression invites to consider a polarization in z:

$$E_g(z,w) = \exp(-\frac{1}{\pi}\int_{\mathbf{C}} \frac{g(\zeta)dA(\zeta)}{(\zeta - z)(\overline{\zeta} - \overline{w})}), \quad z, w \in \mathbf{C} \setminus \mathrm{supp}(g). \tag{6.1}$$

The resulting function $E_g(z,w)$ is analytic in z and antianalytic in w, outside the support of the function g. Note that the integral converges for every pair $(z,w) \in \mathbf{C}^2$ except the diagonal $z = w$. Moreover, assuming by convention $\exp(-\infty) = 0$, a simple application of Fatou's Theorem reveals that the function $E_g(z,w)$ extends to the whole $\mathbf{C}^2$ and it is separately continuous there. Details about these and other similar computations are contained in [23].

As before, the exponential transform contains, in its power expansion at infinity, the moments

$$a_{mn} = a_{mn}(g) = \int_{\mathbf{C}} z^m \overline{z}^n g(z) dA(z), \quad m, n \geq 0.$$

According to Riesz Theorem these data determine g. We will denote the resulting series by:

$$\exp[\frac{-1}{\pi}\sum_{m,n=0}^{\infty} \frac{a_{mn}}{z^{n+1}\overline{w}^{m+1}}] = 1 - \sum_{m,n=0}^{\infty} \frac{b_{mn}}{z^{n+1}\overline{w}^{m+1}}. \tag{6.2}$$

The exponential transform of a uniformly distributed mass on a disk is simple, and in some sense special, this being the building block for more complicated domains. A direct elementary computation leads to the following formulas for the unit disk $\mathbf{D}$, cf. [23]:

$$E_{\mathbf{D}}(z,w) = \begin{cases} 1 - \frac{1}{z\overline{w}}, & z, w \in \overline{\mathbf{D}}^c, \\ 1 - \frac{\overline{z}}{\overline{w}}, & z \in \mathbf{D},\ w \in \overline{\mathbf{D}}^c, \\ 1 - \frac{w}{z}, & w \in \mathbf{D},\ z \in \overline{\mathbf{D}}^c, \\ \frac{|z-w|^2}{1 - z\overline{w}}, & z, w \in \mathbf{D}. \end{cases}$$

Remark that $E_{\mathbf{D}}(z) = E_{\mathbf{D}}(z, z)$ is a rational function and its value for $|z| > 1$ is $1 - \frac{1}{|z|^2}$. The coefficients b_{mn} of the exponential transform are in this case particularly simple: $b_{00} = 1$ and all other values are zero.

Once more, an additional structure of the exponential transform in two variables comes from operator theory. More specifically, for every measurable function $g : \mathbf{C} \longrightarrow [0, 1]$ of compact support there exists a unique irreducible, linear bounded operator $T \in L(H)$ acting on a Hilbert space H, with rank-one self-commutator $[T^*, T] = \xi \otimes \xi = \xi\langle\cdot, \xi\rangle$, which factors E_g as follows:

$$E_g(z, w) = 1 - \langle (T^* - \overline{w})^{-1}\xi, (T^* - \overline{z})^{-1}\xi\rangle, \quad z, w \in \mathrm{supp}(g)^c. \tag{6.3}$$

As a matter of fact, with a proper extension of the definition of localized resolvent $(T^* - \overline{w})^{-1}\xi$ the above formula makes sense on the whole $\mathbf{C}^2$. The function g is called the *principal function* of the operator T. It was introduced in a seminal paper of J. D. Pincus [40], and has been studied by many researchers among whom we cite: Berger, Carey, Clancey, Helton, Howe, Xia; the monographs [37] and [63] treat various aspects of this theory. The bijective correspondence between classes $g \in L^\infty_{\mathrm{comp}}(\mathbf{C})$, $0 \le g \le 1$ and irreducible operators T with rank-one self-commutator was exploited in [44], [45] for solving the L-problem of moments in two variables.

For the aims of this article, the analogy between the principal function and the phase shift is worth mentioning in more detail. More precisely, if $B = A - \xi \otimes \xi$ is a rank-one perturbation of a bounded self-adjoint operator $A \in L(H)$, then for every polynomial $p(z)$, Krein's *trace formula* holds:

$$\mathrm{tr}[p(B) - p(A)] = \int_{\mathbf{R}} p'(t) f_{A,B}(t) dt,$$

where $f_{A,B}$ is the corresponding phase-shift function, [34].

For an operator T possessing rank-one self-commutator $[T^*, T] = \xi \otimes \xi$ and principal function g one can define a non-commutative functional calculus $p(T^*, T)$ with polynomials $p(w, z)$ by putting all powers of T^* to the left of those of T, in each monomial. Then for an arbitrary pair of polynomials $p(\overline{z}, z), q(\overline{z}, z)$ a similar trace formula holds:

$$\mathrm{tr}[p(T^*, T), q(T^*, T)] = \frac{1}{\pi} \int_{\mathbf{C}} \Big(\frac{\partial p}{\partial \overline{z}} \frac{\partial q}{\partial z} - \frac{\partial p}{\partial z} \frac{\partial q}{\partial \overline{z}}\Big) g(z) dA(z).$$

This trace formula was discovered by Helton and Howe [30] and it was the source of modern advances in the cohomology theory of operator algebras.

To draw a conclusion from this comparison, the principal function of a hyponormal operator is a two dimensional analogue of the phase shift of a perturbation of self-adjoint operators. In both cases the inverse spectral problem $f \mapsto (A, \xi)$, respectively $g \mapsto T$, solves Markov's L-problem of power moments, in dimensions 1, respectively 2.

Let $g : \mathbf{C} \longrightarrow [0,1]$ be a measurable function and let $E_g(z,w)$ be its polarized exponential transform. We retain from the above discussion the fact that the kernel:

$$1 - E_g(z,w), \quad z, w \in \mathbf{C},$$

is positive definite. Therefore the distribution $H_g(z,w) = -\frac{\partial}{\partial \overline{z}} \frac{\partial}{\partial w} E_g(z,w)$ has compact support and it is positive definite, in the sense:

$$\int_{\mathbf{C}^2} H_g(z,w)\phi(z)\overline{\phi(w)} dA(z) dA(w) \geq 0, \quad \phi \in C^\infty(\mathbf{C}).$$

If g is the characteristic function of a bounded domain $\Omega \subset \mathbf{C}$, then it is elementary to see that the distribution $H_\Omega(z,w) = H_g(z,w)$ is given on $\Omega \times \Omega$ by a smooth, jointly integrable function which is analytic in $z \in \Omega$ and antianalytic in $w \in \Omega$, see [23].

In particular, this gives the useful representation:

$$E_\Omega(z,w) = 1 - \frac{1}{\pi^2} \int_{\Omega^2} \frac{H_\Omega(u,v) dA(u) dA(v)}{(u-z)(\overline{v} - \overline{w})}, \quad z, w \in \overline{\Omega}^c, \tag{6.4}$$

where the kernel H_Ω is positive definite in $\Omega \times \Omega$.

The example of the disk considered in this section suggests that the exterior exponential transform of a bounded domain $E_\Omega(z,w)$ may extend analytically in each variable inside Ω. This is true whenever $\partial\Omega$ is real analytic smooth. In this case there exists an analytic function S defined in a neighborhood of $\partial\Omega$, with the property:

$$S(z) = \overline{z}, \quad z \in \partial\Omega.$$

The anticonformal local reflection with respect to $\partial\Omega$ is then the map $z \mapsto \overline{S(z)}$; for this reason $S(z)$ is called the *Schwarz function* of the real analytic curve $\partial\Omega$, see [58]. Let ω be a relatively compact subdomain of Ω, with smooth boundary, too, and such that the Schwarz function $S(z)$ is defined on a neighborhood of $\Omega \setminus \omega$. A formal use of Stokes' Theorem in (6.4) yields:

$$1 - E_\Omega(z,w) = \frac{1}{4\pi^2} \int_{\partial\Omega} \int_{\partial\Omega} H_\Omega(u,v) \frac{\overline{u} du}{u - z} \frac{v d\overline{v}}{\overline{v} - \overline{w}} =$$

$$\frac{1}{4\pi^2} \int_{\partial\omega} \int_{\partial\omega} H_\Omega(u,v) \frac{\overline{u} du}{u - z} \frac{v d\overline{v}}{\overline{v} - \overline{w}}. \tag{6.5}$$

But the latter integral is analytic/antianalytic for $z, w \in \overline{\omega}^c$. A little more work with the above Cauchy integrals leads to the following remarkable formula for the analytic extension of $E_\Omega(z,w)$ from $z, w \in \overline{\Omega}^c$ to $z, w \in \overline{\omega}^c$, see [26]:

$$F(z,w) = \begin{cases} E(z,w), & z,w \in \Omega^c, \\ (z - \overline{S(w)})(S(z) - \overline{w})H_\Omega(z,w), & z,w \in \Omega \setminus \overline{\omega}. \end{cases}$$

The study outlined above of the analytic continuation phenomenon of the exponential transform $E_\Omega(z,w)$ led to a proof of a priori regularity of boundaries of domains which admit analytic continuation of their Cauchy transform, [23]. The most general result of this type was obtained by different means by Sakai [54], [55] . We simply state the theorem.

Theorem(Sakai). *Let Ω be a bounded planar domain with the property that its Cauchy transform*

$$\hat{\chi}_\Omega(z) = \frac{-1}{\pi}\int_\Omega \frac{dA(w)}{w-z}, \quad z \in \overline{\Omega}^c$$

extends analytically across $\partial\Omega$. Then the boundary $\partial\Omega$ is real analytic.

Moreover, Sakai has classified in [55] the possible singular points of the boundary of such a domain. For instance angles not equal to 0 or π cannot occur on the boundary.

§7. Extremal domains in two variables

If we would infer from the one-variable picture a good class of extremal domains for Markov's L-problem in two variables we would choose the disjoint unions of disks, as immediate analogs of disjoint unions of intervals, cf. Proposition 4.1. In reality, the nature of the complex plane is more complicated, but again, fortunately for our subject, another correspondence with an external area of function theory can be established.

A bounded domain Ω of the complex plane is called a *quadrature domain* if there exists a finite set of points $a_1, a_2, \ldots, a_d \in \Omega$, and real weights $c_1, c_2, \ldots, c_d$, with the property:

$$\int_\Omega f(z)dA(z) = c_1 f(a_1) + c_2 f(a_2) + \ldots + c_d f(a_d), \quad f \in AL^1(\Omega), \quad (7.1)$$

where the latter denotes the space of all integrable analytic functions in Ω. In case some of the above points coincide, a derivative of f can correspondingly be evaluated.

For example a disk is a quadrature domain, by Gauss Mean Value Theorem:

$$\int_{D(a,r)} f(z)dA(z) = \pi r^2 f(a), \quad f \in AL^1(D(a,r)).$$

The class of quadrature domains was introduced by Aharonov and Shapiro [2] in connection with an extremal problem in conformal mapping; separately and about the same time, Davis [13] and Sakai [52], [53] have studied the same class of planar domains. Later on the subject of quadrature domains has flourished due to other remarkable links with holomorphic partial differential equations [58], Riemann surfaces, potential theory [52], [20], [22], fluid mechanics [58] and operator theory [44], [64], [65], [66].

Before continuing our investigation of the exponential transform we list a few simple properties of quadrature domains. A fine reference, containing many insights and historical remarks, is Shapiro's book [58].

Let Ω be a quadrature domain satisfying (7.1). Then its Cauchy transform is rational:

$$\frac{-1}{\pi}\int_\Omega \frac{dA(w)}{w-z} = \sum_{i=1}^{d} \frac{c_i}{a_i - z}, \ z \in \overline{\Omega}^c. \tag{7.2}$$

Thus, according to Sakai's Theorem, the boundary of Ω is real analytic. Actually, much more is known: the boundary of Ω is real algebraic, given by an irreducible equation of a special form:

$$Q(z,w) = p_d(z)\overline{p_d(w)} - \sum_{j=0}^{d-1} p_j(z)\overline{p_j(w)}, \tag{7.2}$$

where

$$p_d(z) = (z-a_1)(z-a_2)\dots(z-a_d),$$

and $p_j(z)$ is a polynomial of degree j for all $j,\ 0 \le j < d$. For proofs see [20] and [24].

The function

$$S(z) = \overline{z} - \hat{\chi}_\Omega(z) + \sum_{i=1}^{d} \frac{c_i}{a_i - z}, \quad z \in \Omega,$$

is the Schwarz function of the boundary of Ω. Indeed, S is continuous on $\overline{\Omega}$, equal to $\overline{z}$ on $\partial\Omega$ (by (7.2)) and it is analytic on $\Omega \setminus \{a_1, \dots, a_d\}$.

Conversely, if a domain admits a Schwarz function S for its boundary, and S is meromorphic inside, then it is a quadrature domain [58].

A bounded simply connected region is a quadrature domain if and only if it is the image of the unit disk by a rational conformal map, [2], [58]. Thus the cardioid is a quadrature domain with a double node. Quadrature domains with three distinct nodes can be multiply connected.

Among the non-connected quadrature domains we recognize all disjoint unions of disks. Note that these are sufficient to exhaust in area measure any planar domain. An even stronger result holds: quadrature domains are dense among all planar domains, with respect to the Hausdorff metric, see [20].

Let Ω be a bounded planar domain with moments

$$a_{mn} = a_{mn}(\Omega) = \int_\Omega z^m \overline{z}^n dA(z), \quad m, n \geq 0.$$

The exponential transform (6.2) produces the sequence of numbers $b_{mn} = b_{mn}(\Omega), \quad m, n \geq 0$. In virtue of the factorization (6.3),

$$b_{mn} = \langle T^{*m}\xi, T^{*n}\xi \rangle, \quad m, n \geq 0.$$

Hence the matrix $(b_{mn})_{m,n=0}^{\infty}$ turns out to be non-negative definite. The following result identifies a part of the extremal solutions of the L-problem of moments as the class of quadrature domains.

Theorem (7.1). *A bounded planar domain Ω is a quadrature domain if and only if there exists a positive integer $d \geq 1$ with the property*

$$\det(b_{mn}(\Omega))_{m,n=0}^{d} = 0.$$

For a proof see [44]. The vanishing condition in the statement is equivalent to the fact that the span H_d of the vectors $\xi, T^*\xi, T^{*2}\xi, \ldots$ is finite dimensional (in the Hilbert space where the associated hyponormal operator T acts). Thus, if Ω is a quadrature domain with corresponding hyponormal operator T, and T_d is the compression of T to the d-dimensional subspace H_d, then:

$$E_\Omega(z, w) = 1 - \langle (T_d^* - \overline{w})^{-1}\xi, (T_d^* - \overline{z})^{-1}\xi \rangle, \quad z, w \in \overline{\Omega}^c.$$

In particular this proves that the exponential transform of a quadrature domain is a rational function. As a matter of fact a more precise statement can easily be deduced.

Corollary (7.2). *Let Ω be a quadrature domain with data (7.1). Then*

$$E_\Omega(z, w) = \frac{Q(z, w)}{P(z)\overline{P(w)}}, \quad z, w \in \overline{\Omega}^c.$$

Thus the exponential transform of a quadrature domain contains explicitly the irreducible polynomial Q which defines the boundary and the polynomial P which vanishes at the quadrature nodes. By putting together all these remarks we obtain a strikingly similar picture to that of a single variable (see Proposition 4.1). More specifically, if Ω is a quadrature domain with d nodes, with data (7.1) and associated hyponormal operator T, then:

$$E_\Omega(z,w) = \frac{Q(z,w)}{P(z)\overline{P(w)}} =$$

$$1 - \langle (T_d^* - \overline{w})^{-1}\xi, (T_d^* - \overline{z})^{-1}\xi\rangle =$$

$$\frac{1}{\pi^2}\sum_{i,j=1}^{d} H_\Omega(a_i,a_j)\frac{c_i}{a_i - z}\frac{\overline{c_j}}{\overline{a_j}-\overline{w}}, \quad z,w \in \overline{\Omega}^c.$$

In particular we infer, assuming that all nodes are simple:

$$-\pi^2\frac{Q(a_i,a_j)}{P'(a_i)\overline{P'(a_j)}} = c_i\overline{c_j}H_\Omega(a_i,\overline{a_j}), \quad 1 \le i,j \le d.$$

For details see [23], [24], [25], [26].

The interplay between these additive, multiplicative and Hilbert space decompositions of the exponential transform gives an exact reconstruction algorithm of a quadrature domain from its moments. The next section will be devoted to this algorithm.

Before ending the present section we consider a simple illustration of the above formulas. Let $\Omega = \cup_{i=1}^d D(a_i, r_i)$ be a union of d pairwise disjoint disks. This is a quadrature domain with data:

$$P(z) = (z-a_1)\dots(z-a_d),$$

$$Q(z,w) = [(z-a_1)(\overline{w}-\overline{a_1}) - r_1^2]\dots[(z-a_d)(\overline{w}-\overline{a_d}) - r_d^2].$$

The associated matrix T_d is also computable, involving a sequence of square roots of matrices, but we do not need here its precise form. Whence the exponential transform is, for large values of $|z|, |w|$:

$$E_\Omega(z,w) = \prod_{i=1}^{d}[1 - \frac{r_i^2}{(z-a_i)(\overline{w}-\overline{a_i})}] =$$

$$1 + \sum_{i,j=1}^{d}\frac{Q(a_i,\overline{a_j})}{P'(a_i)\overline{P'(a_j)}}\frac{r_i}{a_i - z}\frac{r_j}{\overline{a_j}-\overline{w}}.$$

The essential positive definiteness (6.3) of the exponential transform of an arbitrary domain can be deduced, via an approximation argument, from the positivity of the matrix $(-Q(a_i,\overline{a_j}))_{i,j=1}^d$, where Q is the defining equation of a disjoint union of disks. We note that $(-Q(a_i,\overline{a_j}))_{i,j=1}^d \ge 0$ is only a necessary condition for the disks $D(a_i,r_i)$, $1 \le i \le d$, to be disjoint. Exact computations for $d = 2$ immediately show that this matrix can remain positive definite even the two disks overlap a little. However, if two disks overlap, then, by adding an external disk, even far away, this prevents the new 3×3 matrix to be positive definite.

§8. Applications to shape reconstruction

In complete analogy with the analysis and approximation of Markov functions we explain below how one can use the fine structure of the exponential transform of a planar domain for its reconstruction from a finite segment of its moments.

Let $(a_{mn})_{m,n=0}^d$ be the moment sequence of a measurable function of compact support $g : \mathbf{C} \longrightarrow [0,1]$ and let us consider its formal exponential transform:

$$\exp[\frac{-1}{\pi} \sum_{m,n=0}^{d} \frac{a_{mn}}{z^{n+1}\overline{w}^{m+1}}] = 1 - \sum_{m,n=0}^{\infty} \frac{b_{mn}}{z^{n+1}\overline{w}^{m+1}}. \tag{8.1}$$

A characterization of all sequences (b_{mn}) which can occur in this process is discussed in Chapter XII of [37] or [44].

In view of the remarks outlined in the previous section we have the following algorithm (of identification of a quadrature domain from its moments).

1. Assume that $\det(b_{mn})_{m,n=0}^d = 0$ and that d is the minimal integer with this property, that is $\det(b_{mn})_{m,n=0}^{d-1} \neq 0$. Solve the system:

$$\sum_{m=0}^{d} b_{mn}c_m = 0, \ 0 \le n \le d,$$

with the normalization $c_d = 1$.

2. Consider the polynomial $P(z) = c_d z^d + c_{d-1}z^{d-1} + \ldots + c_0$ and isolate from the following product the polynomial part Q:

$$P(z)\overline{P(w)} \sum_{m,n=0}^{d} \frac{b_{mn}}{z^{n+1}\overline{w}^{m+1}} = Q(z,w) + O(\frac{1}{z}, \frac{1}{\overline{w}}).$$

3. The function g equals, up to a null set, the characteristic function of the quadrature domain $\Omega = \{z \in \mathbf{C};\ Q(z,z) < 0\}$.

The analogy with the one variable algorithm presented in Section 5 is now transparent. Again we exemplify by a simple case.

Start with the data:

$$a_{00} = \pi r^2,\ a_{01} = \overline{a_{10}} = \pi a r^2,\ a_{11} = \pi |a|^2 r^2 + \pi r^4, \tag{8.2}$$

where $r > 0$ and $a \in \mathbf{C}$. Its exponential transform is:

$$\exp[-\frac{r^2}{z\overline{z}} - \frac{ar^2}{z^2\overline{z}} - \frac{\overline{a}r^2}{z\overline{z}^2} - \frac{|a|^2r^2 + r^4}{z^2\overline{z}^2}] =$$

$$1 - \frac{r^2}{z\overline{z}} - \frac{ar^2}{z^2\overline{z}} - \frac{\overline{a}r^2}{z\overline{z}^2} - \frac{|a|^2r^2 + r^4/2}{z^2\overline{z}^2} + \frac{1}{2}\frac{z^4}{z^2\overline{z}^2} + \ldots,$$

so that

$$b_{00} = r^2, \; b_{01} = \overline{b_{10}} = ar^2, \quad b_{11} = |a|^2r^2.$$

The determinant of the associated matrix vanishes:

$$\det(b_{ij})_{i,j=0}^1 = |a|^2r^4 - |a|^2r^4 = 0$$

and the vector $(-a, 1)^T$ is annihilated by this matrix. Therefore

$$P(z) = z - a,$$

and

$$(z-a)(\overline{w}-\overline{a})[1 - \frac{r^2}{z\overline{z}} - \frac{ar^2}{z^2\overline{z}} - \frac{\overline{a}r^2}{z\overline{z}^2} - \frac{|a|^2r^2}{z^2\overline{z}^2} + \ldots] =$$

$$(z-a)(\overline{w}-\overline{a}) - r^2 + O(\frac{1}{z}, \frac{1}{\overline{w}}).$$

Thus the domain with the prescribed moments (8.2) is the disk of equation $|z-a|^2 - r^2 < 0$.

The above algorithm applies as well to non-quadrature domains. The convergence of the Padé type approximation is more involved and requires an analysis of the representations (6.3) and (6.4). We sketch only the main idea, more details being contained in [46].

Let A be a bounded, linear operator with cyclic vector ξ, acting on a Hilbert space H. The series

$$1 - \langle (A-z)^{-1}\xi, (A-w)^{-1}\xi \rangle = 1 - \sum_{i,j=0}^{\infty} \frac{b_{ij}}{z^{i+1}\overline{w}^{j+1}}, \quad |z|, |w| > \|A\|, \tag{8.3}$$

is convergent and has the structure (6.3) of an exponential transform of a function $g \in L^\infty_{\text{comp}}(\mathbf{C}),\ 0 \le g \le 1$. Let H_d be the finite dimensional vector subspace of H generated by $\xi, A\xi, \ldots, A^{d-1}\xi$ and let π_d be the orthogonal projection onto H_d. We will assume that $\dim H_d = d$ for all $d \ge 1$, that is the operator A does not have finite rank. Let $A_d = \pi_d A \pi_d$ be the compression of A to H_d.

Lemma (8.1). *With the above notation, the minimal polynomial* $p_d(z) = z^d + c_{d-1}z^{d-1} + \ldots + c_0$ *of the matrix* A_d *satisfies*

$$\min \sum_{i,j=0}^{d} \gamma_i b_{ij} \overline{\gamma_j} = \sum_{i,j=0}^{d} c_i b_{ij} \overline{c_j},$$

where the minimum is taken among all sequences of complex numbers $\gamma_0, \ldots, \gamma_{d-1},\ \gamma_d = 1$.

Next we find that there exists a polynomial $q_{d-1}(z, \overline{w})$ of degree $d-1$ in each variable, satisfying

$$p_d(z)\overline{p_d(w)}[\sum_{i,j=0}^{\infty} \frac{b_{ij}}{z^{i+1}\overline{w}^{j+1}}] =$$

$$q_{d-1}(z, \overline{w}) + \frac{\beta_{dd}}{z^{d+1}\overline{w}^{d+1}} + \sum_{\max(i,j)>d} \frac{\beta_{ij}}{z^{i+1}\overline{w}^{j+1}}, \tag{8.4}$$

with some constants β_{ij}.

That this is a right analogue of the diagonal Padé approximant one can see from the identification:

$$\frac{q_{d-1}(z, \overline{w})}{p_d(z)\overline{p_d(w)}} = \langle (A_d - z)^{-1}\xi, (A_d - w)^{-1}\xi \rangle.$$

This is in complete analogy again with the computations performed in Section 5.

We recall that the *numerical range* of the operator A is the set $W(A) = \{\langle Ax, x\rangle;\ \ \|x\| = 1\}$. By a classical result of Hausdorff and Toeplitz, the numerical range $W(A)$ is a convex set which contains in its closure the spectrum of A. In virtue of von-Neumann's inequality ([47]):

$$\|(A - z)^{-1}\| \leq \frac{1}{\operatorname{dist}(z, W(A))}, \quad z \in \overline{W(A)}^c,$$

we deduce as in Section 5 above that, for every vector $x \in H$:

$$\lim_{d\to\infty} \|(A_d - z)^{-1}x - (A - z)^{-1}x\| = 0,$$

uniformly on compact subsets of $\mathbf{C} \setminus \overline{W(A)}$. This leads to the following generalization of Markov's Theorem.

Theorem (8.2). *The diagonal Padé approximation* $1 - \frac{q_{d-1}(z,\overline{w})}{p_d(z)\overline{p_d(w)}}$ *converges uniformly to the series (8.3), on every compact set which is disjoint of the closed numerical range of the operator A.*

When working with the moments of a characteristic function $g = \chi_\Omega$ of a domain with smooth, real analytic boundary, the above convergence can be extended in general across $\partial\Omega$. The rate of convergence turns out to be exponential and depending on the logarithmic capacity of the set ω up to which the Schwarz function of $\partial\Omega$ analytically extends(cf. (6.5)) and the distance to this set. More details can be found in [46].

A series of numerical experiments [19], [27] have validated the above reconstruction algorithm. For domains with corners the numerical evidence is at this point stronger than the theoretical results.

§9. The exponential transform in n dimensions

The generalization of the exponential transform and its related reconstruction algorithms from one and two variables to more than two variables can be considered still in an incipient stage. So far it is known that the exponential transform of a bounded domain in $\mathbf{R}^n$ is sub-harmonic (it behaves like the equilibrium potential) and has a controlled decay to zero on smooth portions of the boundary. On simple domains, such as quadratic domains and polyhedra it is very close to the better studied low dimensional cases. Although the positivity properties of the exponential transform in $\mathbf{R}^n$ do not lack, a successful Hilbert space factorization (as for $n = 1, 2$) is not known in general. The present section summarizes some recent results proved in [26].

Let $\Omega \subset \mathbf{R}^n, n \geq 1$, be a bounded domain with smooth boundary. We denote by $|S^{n-1}|$ the $(n-1)$-volume of the unit sphere in $\mathbf{R}^n$, so that $|S^0| = 2$ by convention. The associated *exponential transform* is:

$$E_\Omega(x) = \exp[\frac{-2}{|S^{n-1}|} \int_\Omega \frac{dy}{|y-x|^n}].$$

Assuming as before $\exp(-\infty) = 0$, $E_\Omega(x)$ is a continuous function on $x \in \mathbf{R}^n$, and $E_\Omega(\infty) = 1$. According to Riesz Theorem, the germ at infinity of E_Ω determines Ω.

A few geometric interpretations are available from the very definition of E_Ω. Indeed, let $n \geq 2$ and let $d\theta(x)$ stand for the solid angle form in $\mathbf{R}^n$ (see for instance [16]). Then

$$E_\Omega(x) = \exp[\frac{-2}{|S^{n-1}|} \int_{\partial\Omega} \log|x-y| d\theta(y-x)], \quad x \in \overline{\Omega}^c. \tag{9.1}$$

Thus $E_\Omega(x)$ can be interpreted as the geometric mean of the distance function $|x-y|$, $y \in \partial\Omega$, with respect to the solid angle measure $d\theta(y-x)$. For a proof see Proposition 3.1 of [26].

This formula suggests a companion inner exponential transform, exactly as we have encountered in the two dimensional case. Specifically:

$$H_\Omega(x) = \exp[\frac{-2}{|S^{n-1}|}\int_{\partial\Omega} \log|x-y| d\theta(y-x)], \quad x \in \Omega.$$

Note that, for $n \geq 3$, the Newtonian potential of the volume Ω and its associated field have very similar formulas:

$$U^\Omega(x) = \frac{1}{2(n-2)|S^{n-1}|}\int_{\partial\Omega} |x-y|^2 d\theta(y-x),$$

and

$$\nabla U^\Omega(x) = \frac{1}{|S^{n-1}|}\int_{\partial\Omega} (x-y) d\theta(y-x), \quad x \notin \partial\Omega.$$

These quantities can also be expressed in close form in terms of the spherical means of the domain Ω.

Note also, on the easy remarks side, that E_Ω depends multiplicatively of Ω. That is, if Ω_1 and Ω_2 are two disjoint domains and $\Omega = \Omega_1 \cup \Omega_2$, then

$$E_\Omega(x) = E_{\Omega_1}(x) E_{\Omega_2}(x), \quad x \in \Omega^c,$$

while

$$H_\Omega(x) = H_{\Omega_1}(x) E_{\Omega_2}(x), \quad x \in \Omega_1.$$

A few examples are available by direct computation (see Section 4 of [26]). For instance, in the case of a half-space $H = \{x_n < 0\}$ we obtain:

$$\exp[\frac{-2}{|S^{n-1}|}\int_{\partial H} \log|x-y| d\theta(y-x)] = \begin{cases} a_n x_n, & x_n > 0, \\ -(a_n x_n)^{-1}, & x_n < 0. \end{cases}$$

The above constant a_n depends only on n. Of course the integral in the exponential transform of the half-space H is divergent. The above boundary integrals being relevant only after a truncation of the corresponding volume integrals at infinity.

With a similar normalization one finds the dimensionless formula:

$$E_{\{a<x_n<b\}}(x) = \begin{cases} \frac{x_n - b}{x_n - a}, & x_n > b, \\ \frac{x_n - a}{x_n - b}, & x_n < a. \end{cases}$$

In general the exponential transform is well behaved with respect to cylinders:

$$E_{\Omega\times\mathbf{R}^m}(x,y) = E_\Omega(x).$$

The above explicit formulas for the exponential transform of a region bounded by hyperplanes can be put into a more invariant form, at least for convex polyhedra. The *ridge* of a convex polyhedron Ω is the set of all interior points with the property that the distance to $\partial\Omega$ is attained at two or more distinct points. For instance the ridge of a triangle is formed by its bisectors.

Proposition (9.1). *Let Ω be a convex polyhedron with ridge R. The exponential transform E_Ω can be analytically continued from the exterior of Ω to R, by the explicit formula:*

$$F(x) = \begin{cases} E_\Omega(x), & x \in \Omega^c, \\ -a_n^2 \mathrm{dist}(x, \partial\Omega)^2 H_\Omega(x), & x \in \Omega \setminus R. \end{cases}$$

Note that this real analytic extension vanishes of the first order on the facets of Ω, exactly as the analysis of a single hyperplane has shown. This observation might be useful for reconstructing (numerically) a polyhedron from some external data.

Continuing the list of examples, a quadratic surface reveals a very simple interior transform. Indeed, assume that $\Omega = \{x \in \mathbf{R}^n;\ q(x) < 0\}$, where q is a quadratic polynomial with positive definite leading homogeneous part (so that Ω is bounded). Then there exists a negative constant C with the property:

$$H_\Omega(x) = \frac{C}{q(x)}, \quad x \in \Omega.$$

As for the unit ball, the following inductive formula holds, this time for the exterior transform:

$$E_1(r) = \frac{r-1}{r+1}, \quad E_2(r) = 1 - \frac{1}{r^2},$$

$$E_n(r) = E_{n-2}(r)\exp[\frac{2}{(n-2)r^{n-2}}], \quad n \geq 3,$$

where the simplified notation was used: $r = |x|,\ E_n(r) = E_{\{|x|<1\}}(r)$. Thus, $E_3(r) = \frac{r-1}{r+1}\exp[\frac{2}{r}]$ is no more a rational function.

As for the decay rate of the exponential transform towards the boundary of the domain, the following general result can be proved.

Proposition (9.2). *Let Ω be a bounded domain in $\mathbf{R}^n$ with C^2 smooth boundary. There are positive constants C_1, C_2 depending on the geometry of the domain, such that:*

$$C_1 \mathrm{dist}(x, \Omega) \leq E_\Omega(x) \leq C_2 \mathrm{dist}(x, \Omega),$$

for x exterior and close to Ω.

We have seen that, in two dimensions, the function $1 - E_\Omega$ can be polarized and is positive definite. Unfortunately this strong positivity does not hold in higher dimensions. Instead, one can only prove the following result.

Theorem (9.3). *Let $\Omega \subset \mathbf{R}^n$, $n \geq 3$, be a bounded domain. Then:*

$$\Delta(1 - E_\Omega(x))^{(n-2)/n} \geq 0, \quad x \in \overline{\Omega}^c.$$

The proof of the subharmonicity of $1 - E$ can be found in [26]. The above stronger result is due to Tkachev [61]. It shows that the function $1 - E$ is similar to the equilibrium potential u of the set $\overline{\Omega}$, normalized to be equal to 1 on $\partial\Omega$.

Indeed, for large values of $|x|$:

$$u(x) = \frac{\operatorname{cap}(\overline{\Omega})}{(n-2)|S^{n-1}|} \frac{1}{|x|^{n-2}} + O(\frac{1}{|x|^{n-1}}),$$

where "cap" stands for the Newtonian capacity, while

$$1 - E_\Omega(x) = \frac{2|\Omega|}{|S^{n-1}|} \frac{1}{|x|^n} + O(\frac{1}{|x|^{n+1}}),$$

where $|\Omega|$ is the volume of Ω.

The fractional power of $1 - E_\Omega$ makes the two asymptotics at infinity comparable. Assuming that the boundary of Ω is smooth, we also know that both u and $1 - E_\Omega$ tend to 1 towards $\partial\Omega$. Due to the subharmonicity of $(1 - E_\Omega(x))^{(n-2)/n}$ we obtain the estimate

$$(1 - E_\Omega(x))^{(n-2)/n} \leq u(x), \quad x \in \Omega^c.$$

By comparing the leading terms at infinity we find:

$$[\frac{2|\Omega|}{|S^{n-1}|}]^{(n-2)/n} \leq \frac{\operatorname{cap}(\overline{\Omega})}{(n-2)|S^{n-1}|}.$$

An approximation of an arbitrary domain by inner domains with smooth boundary leads to the following inequality.

Corollary (9.4). *Let $\Omega \subset \mathbf{R}^n$, $n \geq 3$, be a bounded domain satisfying $\Omega = \operatorname{int}\overline{\Omega}$. Then*

$$(n-2)2^{(n-2)/n}|S^{n-1}|^{2/n}|\Omega|^{(n-2)/2} \leq \operatorname{cap}(\overline{\Omega}).$$

This inequality is an analogue of Ahlfors and Beurling estimate of the logarithmic capacity of a planar domain $\Omega \subset \mathbf{R}^2$:

$$\pi^{-1/2}|\Omega|^{1/2} \leq \operatorname{cap}(\overline{\Omega}),$$

see [36], [26].

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Mihai Putinar,
Mathematics Department,
University of California,
Santa Barbara, CA 93106
`mputinar@math.ucsb.edu`