# On multivariable Fejér inequalities

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**Abstract**

A non-negative pluriharmonic polynomial $\Re p(z)$ on the unit ball of $\mathbb{C}^n$ is used as a weight against the rotationally invariant measure on the unit sphere. The resulting Hardy space carries the canonical n-tuple $S$ of multiplication by the coordinate functions. By means of compressions of $S$ to co-analytically invariant subspaces, and known estimates of the numerical radius of a nilpotent matrix we obtain bounds for the coefficients of $p$, in terms of the arithmetic mean and degree of $p$, and dimension $n$.

MSC 2000: 42B05, 47A12, 31C10

## 1 Introduction

The following remarkable inequality was discovered by Fejér in 1910 and later published in [4, 5].

**Theorem 1.1** *If* $h(\theta) = \sum_{k=-d}^{k=d} h_k e^{ik\theta} \geq 0$ *for* $\theta \in [0, 2\pi)$ *and* $h_0 = 1$*, then* $|h_1| \leq \cos(\frac{\pi}{d+2})$.

He also showed that the trigonometric polynomial that achieves equality in the above theorem is unique up to the argument of $h_1$. For example, when $d = 2$, the unique trigonometric polynomial such that $|h_1| = \cos(\frac{\pi}{4})$ and $\arg h_1 = \frac{\pi}{4}$ is given by

$$\begin{aligned} h(\theta) &= -\frac{1}{4} i e^{-i2\theta} + \left(\frac{1-i}{2}\right) e^{-i\theta} + 1 + \left(\frac{1+i}{2}\right) e^{i\theta} + \frac{1}{4} i e^{i2\theta} \\ &= \Re[1 + (1+i)z + \frac{1}{2} i z^2]. \end{aligned} \tag{1}$$

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For an analytic polynomial $p(z) = 1+p_1z+\ldots+p_dz^d$ that satisfies $\Re p(z) \geq 0$ for $z \in \partial\mathbb{D}$, Fejér's inequality becomes:

$$|p_1| \leq 2\cos\left(\frac{\pi}{d+2}\right). \tag{2}$$

When inequality (2) is generalized to all coefficients $p_k$, it is known as the Egerváry-Szász inequality: If $p(z) = 1 + p_1z + \ldots + p_dz^d$ satisfies $\Re p(z) \geq 0$ for $z \in \partial\mathbb{D}$, then

$$|p_k| \leq 2\cos\left(\frac{\pi}{\lfloor \frac{d}{k} \rfloor + 2}\right). \tag{3}$$

for $1 \leq k \leq d$. See [3].

The original treatment of (2) was elementary and very elegant. Fejér's main tool was the factorization (known today as Riesz-Fejér's Theorem): $h(\theta) = |Q(e^{i\theta})|^2$, where $Q(z)$ is an analytic polynomial of degree equal to the degree of $h$. See also [9], pp 77-82.

A natural connection between Fejér's inequality and the numerical radius of a nilpotent matrix was recently established by Haagerup and de la Harpe [6]. They proved, using solely elementary methods (positive definite kernels) that:

*If $T$ is a bounded linear operator on a Hilbert space $H$ satisfying $T^{d+1} = 0$ and $\|T\| = 1$, then*

$$w(T) = \sup |\{< T\xi, \xi > \mid \|\xi\| = 1\}| \leq \cos\left(\frac{\pi}{d+2}\right). \tag{4}$$

The extremal operator is shown to be a truncated shift (Jordan block), with an appropriate choice for the vector $\xi$. Furthermore, the authors of [6] point out that the two inequalities, for a positive trigonometric polynomial, and for a nilpotent operator, are equivalent. The numerical quantity $w(T)$ is known as the numerical radius of the operator $T$; see for instance [7].

The aim of the present note is to report a series of extensions of Fejér's and Egerváry-Szász' inequalities to bounded domains of $\mathbb{C}^n$. We will not leave the comfortable assumption of circular symmetry (of Reinhardt domains) which turns all monomials into mutually orthogonal vectors, with respect, say, to a Hardy space or Bergman space metric. Since a non-negative pluriharmonic polynomial $\Re p(z)$ on such a domain cannot in general be represented as a sum of squares of analytic polynomials with bounded degree, Fejér's original proof cannot be adapted to this new framework. Fortunately, the Hilbert space approach of [6] is better suited, and this is the path we follow. For simplicity, we state and prove all of our results on the ball, but the reader will find no additional difficulties in adapting the proofs to any Reinhardt domain, with a rotationally invariant measure, or simply the volume measure instead of $\sigma$.

A few years ago Badea and Cassier [2] investigated operator radii and some generalizations of the Egerváry-Szász inequality in one variable. In addition, Popescu [10] has recently and independently obtained several inequalities of Haagerup-de la Harpe type. These results yielded a multivariable operator-valued Fejér-Egerváry-Szász inequality. The authors thank the referee for several helpful comments and especially for bringing to our attention the works of Badea, Cassier and Popescu.

## 2 Fejér Inequalities on the Ball in $\mathbb{C}^n$

Fix a positive integer $n$. Let $\mathbb{B}$ denote the ball in $\mathbb{C}^n$ of radius 1 and center 0. Let $\mathfrak{I}$ denote the collection of all multiindices of length $n$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index of nonnegative integers and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, then we follow standard notations: $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha! = \alpha_1! \ldots \alpha_n!$, and $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$. The terminology $\alpha \geq \beta$ means that $\alpha_k \geq \beta_k$ for $k = 1, \ldots, n$, while $\alpha > \beta$ implies $\alpha \geq \beta$ but $\alpha \neq \beta$.

Fix a polynomial $p$ of $n$ variables which satisfies $\Re p(z) \geq 0$ for $z \in \partial\mathbb{B}$. Assume $p$ has Taylor's series $p(z) = 1 + \sum_{0<|\alpha|\leq d} p_\alpha z^\alpha$. To obtain a collection of estimates generalizing Fejér's inequalities to polynomials with real part on the unit ball in $\mathbb{C}^n$, we will apply the numerical range bound (4) to a subspace of the Hilbert space $L^2(\mu)$, where $\mathrm{d}\mu(z) = \Re p(z)\mathrm{d}\sigma(z)$ and the measure $\sigma$ is normalized Lebesgue measure on the sphere $\partial\mathbb{B}$. Let $\mathrm{H}^2$ denote the standard Hardy space $\mathrm{H}^2(\sigma)$ on the unit ball; similarly $\mathrm{H}^2(\mu)$ will be the closure in $L^2(\mu)$ of all complex analytic polynomials.

Recall that the monomials $\{z^\alpha\}_{\alpha\in\mathfrak{I}}$ are orthogonal with respect to $\mathrm{d}\sigma$, and that for all $\alpha \in \mathfrak{I}$

$$\|z^\alpha\|_{\mathrm{H}^2}^2 = \int_{\partial\mathbb{B}} |z^\alpha|^2 \,\mathrm{d}\sigma(z) = \frac{\alpha!(n-1)!}{(|\alpha|+n-1)!}.$$

Using these facts, it is easy to show that the coefficients $p_\alpha$ of $p$ are encoded in the moments of the new scalar product:

$$\int_{\partial\mathbb{B}} z^\alpha \overline{z}^\beta \mathrm{d}\mu(z) = \begin{cases} \frac{p_{\beta-\alpha}}{2}\|z^\beta\|_{\mathrm{H}^2}^2 & \text{if } \beta > \alpha \text{ and } |\beta-\alpha| \leq d, \\ \frac{\overline{p}_{\alpha-\beta}}{2}\|z^\alpha\|_{\mathrm{H}^2}^2 & \text{if } \alpha > \beta \text{ and } |\alpha-\beta| \leq d, \\ \|z^\alpha\|_{\mathrm{H}^2}^2 & \text{if } \beta = \alpha, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

See [1, 8] for more properties of positive measures like $\mu$.

Define the analytically invariant subspace $\mathcal{N}_d \subseteq \mathrm{H}^2(\mu)$ by $\mathcal{N}_d = \mathrm{span}\{z^\alpha\}_{|\alpha|>d}$; let $\mathcal{M}_d = \mathcal{N}_d^\perp$,and let $P_d$ denote the orthogonal projection from $\mathrm{H}^2(\mu)$ onto $\mathcal{M}_d$. Set $\epsilon_j$ to denote the multiindex of length $n$ which is 1 in the jth coordinate and 0 in all other coordinates. Define a multiplication operator on $\mathcal{M}_d$ by

$$S_j f = P_d(z_j f) \text{ for all } f \in \mathcal{M}_d.$$

Note that a polynomial $f \in \text{span}\{z^\alpha\}_{|\alpha|\le d}$ may not be orthogonal to $\mathcal{N}_d$. However, its class $[f]$ modulo $\mathcal{N}_d$ can be identified with $P_d f$, which obviously belongs to $\mathcal{M}_d$.

Define the operator n-tuple $S = (S_1, \ldots, S_n)$. By definition, the powers of the $n$-tuple $S$ satisfy

$$S^\gamma [z^\beta] = \begin{cases} [z^{\gamma+\beta}] & \text{whenever } |\gamma+\beta| \le d, \\ 0 & \text{whenever } |\gamma+\beta| \ge d+1. \end{cases}$$

We conclude that $S^\gamma = 0$ if $|\gamma| \ge d+1$ and that $(S^\alpha)^{\lfloor \frac{d}{|\alpha|} \rfloor + 1} = 0$. Therefore the numerical radius of $S^\alpha$ has the following upper bound:

$$w\left(\frac{S^\alpha}{\|S^\alpha\|}\right) \le r_\alpha \tag{6}$$

for all $|\alpha| \le d$, where $r_\alpha = \cos\left(\frac{\pi}{\lfloor \frac{d}{|\alpha|} \rfloor + 2}\right)$. Computing the norm of $S^\alpha$ may not be an easy task. In this direction we state an immediate estimate.

**Proposition 2.1** *Assume $p$ is a polynomial of total degree $d$. If $0 \le |\alpha| \le d$ and $\{j_1, \ldots, j_m\}$ are the indices for which $\alpha_{j_k} \ne 0$ , then the norm $\|S^\alpha\|$ on $\mathcal{M}_d$ satisfies*

$$\|S^\alpha\| \le \Pi_{k=1}^m \left(\frac{\alpha_{j_k}}{|\alpha|}\right)^{\frac{\alpha_{j_k}}{2}}. \tag{7}$$

Proof: The operator $S^\alpha$ is the compression of multiplication by $z^\alpha$ onto $\mathcal{M}_d$; therefore the norm of $S^\alpha$ is bounded above by the supremum of $|z^\alpha|$ on the sphere $\partial\mathbb{B}$. Define $x_{j_k} = |z_{j_k}|$ for $k = 1, \ldots, m$. A Lagrange multiplier argument which maximizes the function $f(x_1, \ldots, x_n) = x_{j_1}^{\alpha_{j_1}} \ldots x_{j_m}^{\alpha_{j_m}}$ subject to the constraint $x_1^2 + \ldots + x_n^2 = 1$ yields the maximum on the right side of inequality (7). □

Remark: Equality will not generally hold in (7).

The following result contains Fejér type inequalities on the ball of $\mathbb{C}^n$.

**Theorem 2.2** *Let $p(z) = \sum_{|\alpha|\le d} p_\alpha z^\alpha$ be a polynomial which satisfies $p(0) = 1$ and $\Re p(z) \ge 0$ for $z \in \partial\mathbb{B}$. Then for any multiindex $\alpha$ with $|\alpha| \le d$,*

$$|p_\alpha| \le \frac{2\|S^\alpha\|}{\|z^\alpha\|^2_{\mathrm{H}^2}} \cos\left(\frac{\pi}{\lfloor \frac{d}{|\alpha|} \rfloor + 2}\right).$$

Proof: Note that the constant functions are in $\mathcal{M}_d$, in particular $\mathbf{1} = P_d \mathbf{1} = \mathbf{1}$. By equation (5), $|\alpha| \le d$ implies $\frac{\overline{p}_\alpha}{2}\|z^\alpha\|^2_{\mathrm{H}^2} = \int_{\partial\mathbb{B}} z^\alpha \, \mathrm{d}\mu = < S^\alpha \mathbf{1}, \mathbf{1} >_{\mathrm{L}^2(\mu)}$ . Next, equation (6) shows that $| < S^\alpha \mathbf{1}, \mathbf{1} >_{\mathrm{L}^2(\mu)} | \le r_\alpha \|S^\alpha\|$, from which the theorem follows. □

Substituting the maximum from Proposition 2.1 into the theorem above yields a concrete inequality; however, it will not be sharp in general. For $n = 1$ we have $2\|S^\alpha\|/\|z^\alpha\|^2_{\mathrm{H}^2} \le 2\|z^\alpha\|_\infty/\|z^\alpha\|^2_{\mathrm{H}^2} \le 2$; hence we return to the classical inequalities (2) and (3). Note that the case $|\alpha| = 1$ is special, due to the next simple observation.

**Corollary 2.3** *If the polynomial* $p(z) = 1 + c_1 z_1 + c_2 z_2 + \ldots + c_n z_n + O(|z|^2)$ *has non-negative real part on the unit ball, then*

$$[|c_1|^2 + \ldots + |c_n|^2]^{1/2} \leq 2\cos\left(\frac{\pi}{d+2}\right).$$

Proof: By a unitary change of variables we can assume $c_2 = c_3 = \ldots = c_n = 0$. Then Fejér's inequality applies to the polynomial of a single variable $p(z_1, 0, \ldots, 0)$, of degree $d' \leq d$. Consequently $|c_1| \leq 2\cos\left(\frac{\pi}{d'+2}\right) \leq 2\cos\left(\frac{\pi}{d+2}\right)$. $\square$

Unlike the one variable case, the polynomials for which equality holds in the above corollary are not unique. For example, if $n = 2$, then equality holds for the continuous family

$$p_t(z_1, z_2) = 1 + (1+i)z_1 + \frac{1}{2} i z_1^2 + t z_2^2$$

where $t \in [0, \frac{1}{2})$. A Lagrange multiplier argument is again needed for proving $\Re p_t \geq 0$ on the ball.

Given an operator-valued multivariable trigonometric polynomial, Popescu's result [10] provides a bound on the euclidean norm of the vector of coefficients corresponding to powers with fixed total degree $k$; that is, it generalizes Corollary 2.3 to operator-valued functions and to any degree less than or equal to the degree of the polynomial.

A detailed analysis of the extremal cases in Theorem 2.2 will be included in a separate article.

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