THE FRIEDRICHS OPERATOR OF A PLANAR DOMAIN

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Dedicated to the memory of S. A. Vinogradov

Abstract. Properties of the Friedrichs operator of a planar domain, originally intended for applications to planar elasticity, are related to the geometry and function theory of the domain. We show that the spectrum of the Friedrichs operator does not, in general, determine the domain, and we provide the additional necessary complete invariants. An analysis of the Friedrichs operator of generalized quadrature domains is carried out at the abstract level and on several examples. The study of Toeplitz operators on the Bergman space of the domain comes naturally into focus.

1. Introduction

K. Friedrichs [7] introduced implicitly a real-linear operator (below denoted F_Ω) on the Bergman space $AL^2(\Omega)$ of a plane domain, more precisely he studied the symmetric (not Hermitean) bilinear form

$$(f, g) \mapsto Re \int_\Omega fg \, dA,$$

which defines, when represented against the L^2-scalar product, the operator F_Ω that we think it is jusitifed to call "the Friedrichs operator" associated to the domain Ω. Above dA stands for the planar Lebesgue measure. The square S_Ω of F_Ω is a complex-linear, Hermitean and non-negative operator on $AL^2(\Omega)$.

These operators are interesting in several respects. First of all, the spectral properties of S_Ω reflect fine-structure properties of the geometric relations in the Hilbert space $L^2(\Omega)$ between its two subspaces consisting respectively of holomorphic and conjugate holomorphic functions. To be more specific, S_Ω is the "gap operator", in the terminology of [11], between these two subspaces. For example, the extreme situation where these spaces are mutually orthogonal (as happens for the so-called null quadrature domains, [19], e.g. the complement of an ellipse) corresponds to $F_\Omega = 0$.

Secondly, these operators allow concise formulations of the boundary value problems for the biharmonic operator that are of importance in the theory of planar elasticity; in particular, in domains for which the operator F_Ω has finite dimensional range (identified in [24] as the class of quadrature domains), the exact solutions of the said boundary value problems are obtained by solving finite systems of linear equations. These aspects seem to have been the main motivation for Friedrichs' work. In elaborating his program, he proved a number of interesting results about these operators, concerning compactness, essential spectrum etc. in their dependence on the underlying domain.

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The aim of this paper is to extend Friedrichs' pioneering work and take up some new aspects motivated by function-theoretic and functional-analytic considerations. We hope in future work to deal with the applications to the biharmonic equation. That there is a very intimate connection can be seen in a striking way from one of our results: F_Ω is injective if and only if the weakly elliptic equation

$$\overline{\partial}^2 u = 0$$

has no non-trivial solution in Ω vanishing on its boundary (note that the symbol of this operator, whose kernel comprises the so-called biholomorphic functions, divides that of the bi-Laplacian).

Another interesting aspect of the Friedrichs operator, which is here taken up in detail for the first time, is its close relation to the so-called (generalized) quadrature domains.

The present paper is organized as follows. Section 2 contains preliminaries and a few simple observations which relate the Friedrichs operator to Toeplitz-Bergman operators and to quadrature domains. Section 3 states Friedrichs' main theorem and its relevance for planar elasticity and an interesting inequality in function theory. In Section 4 we analyze some spectral properties of S_Ω; for instance we show that S_Ω cannot be Fredholm and that the elements in the kernel of S_Ω are remarkable analytic functions. Finally, in the same section we show that there are domains, not affinely equivalent, whose Friedrichs operators have the same spectrum. Section 5 deals with the relation between F_Ω and the generalized quadratures supported by thin subsets of Ω. A restriction operator between the Bergman space and the L^2-space of such a quadrature identity is instrumental in computing or at least having qualitative information about the eigenvalues of S_Ω and the corresponding eigenfunctions. This restriction operator has appeared in [21] and [23] in a functional-theoretic clarification of earlier work by S. Bergman. Section 6 contains some examples of domains on which the Friedrichs operator, or at least its spectrum and eigenfunctions, is within reach.

This paper is intended to be continued by another one, where different aspects related to the Friedrichs operator (already alluded to in the text below) will be developed.

2. Preliminaries

In this section we explain the notation, give some definitions and state a few basic observations concerning the defined objects.

2.1. **Notation.** Let Ω be a domain of the complex plane $\mathbf{C} = \mathbf{R} \oplus i\mathbf{R}$. In general we allow Ω to be unbounded, but we require that its boundary $\partial\Omega$ contains infinitely many points, so that Ω carries non-trivial, square integrable analytic functions. Such assumptions will depend and result from the context. The planar Lebesgue measure will be denoted by dA and the corresponding Lebesgue spaces will be $L^p(\Omega) = L^p(\Omega, dA), 1 \le p \le \infty$. For every $p, 1 \le p \le \infty$, we denote by $AL^p(\Omega)$ the space of complex analytic functions in Ω which belong to $L^p(\Omega)$. In addition, we put:

$$AL^2_0(\Omega) = \{f \in AL^2(\Omega); \int_\Omega f dA = 0\}.$$

Of course $AL^\infty(\Omega) = H^\infty(\Omega)$. In general, $H^p(\Omega)$, $1 \le p \le \infty$, will be the Hardy spaces of a domain Ω with piecewise smooth boundary. The space of analytic functions in an open set Ω will be denoted by $O(\Omega)$; for a closed set K, we denote by $O(K)$ the algebra of germs of analytic functions defined in neighbourhoods of K. The complex conjugation operator:

$$C : L^p(\Omega) \longrightarrow L^p(\Omega), \qquad C(f) = \overline{f}, \quad f \in L^p(\Omega),$$

is an isometry of $L^p(\Omega)$. Thus $CAL^p(\Omega)$ is the space of anti-analytic functions which are p-summable in Ω. We put $\|\phi\|_{p,\Omega}$ for the norm of the space $L^p(\Omega)$.

For a measure or distribution u, its closed support in the sense of distributions will be denoted by $supp(u)$.

The **Bergman space** of the domain Ω is the space $AL^2(\Omega)$. It is a Hilbert space with reproducing kernel $K(z,w) = k_w(z)$, known as the **Bergman kernel** of the domain Ω. The orthogonal projection of $L^2(\Omega)$ onto $AL^2(\Omega)$ will be denoted by P_Ω, or in short, P (the **Bergman projection** of Ω). The scalar product and the norm in $L^2(\Omega)$ will be denoted by $\langle *, * \rangle$, respectively $\| * \|$. Therefore, for any element $f \in AL^2(\Omega)$ and $w \in \Omega$ one has $\langle f, k_w \rangle = f(w)$. The function identically equal to 1 will be denoted by $\mathbf{1}$.

Let $\phi \in L^\infty(\Omega)$ and let

$$T_\phi : AL^2(\Omega) \longrightarrow AL^2(\Omega), \qquad T_\phi(f) = P(\phi f), \quad f \in AL^2(\Omega),$$

be the **Bergman-Toeplitz operator** with symbol ϕ. It is a linear bounded operator of norm $\|T_\phi\| \leq \|\phi\|_{\infty,\Omega}$. The operator $B = T_z$ corresponds to the multiplication on the Bergman space with the complex variable $z \in \Omega$. We call B the **Bergman shift** of Ω.

By convention we assume without explicit mention in the paper, that whenever the constant function $\mathbf{1}$ comes into discussion, the area of Ω is finite, so that $\mathbf{1} \in AL^2(\Omega)$. Also, we speak of the Bergman shift B only when the domain Ω is bounded, so that this operator satisfies $\|B\| < \infty$.

Finally, for an abstract linear bounded operator T acting on a Hilbert space H we denote by $\sigma(T)$ the spectrum of T and by $\sigma_{ess}(T)$ the essential spectrum of T. The identity operator on H is denoted by I.

2.2. **Definitions.** The **Friedrichs operator** of the domain Ω is the anti-linear operator $F = PC : AL^2(\Omega) \longrightarrow AL^2(\Omega)$. Thus, for a function $f \in AL^2(\Omega)$ we have:

$$F(f)(z) = \int_\Omega K(z,w)\overline{f(w)}dA(w). \tag{1}$$

Obviously, $\|F(f)\| \leq \|f\|$, and if the domain Ω is bounded, $F(\mathbf{1}) = \mathbf{1}$. For a pair of functions $f, g \in AL^2(\Omega)$ we note the identities:

$$\langle f, Fg \rangle = \langle g, Ff \rangle = \int_\Omega (fg)dA. \tag{2}$$

The square $S = F^2$ of the Friedrichs operator is $\mathbf{C}$-linear and:

$$\langle f, Sf \rangle = \langle Ff, Ff \rangle.$$

Therefore S is a self-adjoint operator on the Bergman space, satifying $0 \leq S \leq I$.

Actually, the operator S is the "gap operator" between the closed subspaces $AL^2(\Omega)$, $CAL^2(\Omega)$ of $L^2(\Omega)$, see for more details [11], p. 197. Indeed, denoting by $Q = CPC$ the orthogonal projection of $L^2(\Omega)$ onto $CAL^2(\Omega)$, one finds that:

$$S = PCPC|_{AL^2(\Omega)} = PQ|_{AL^2(\Omega)}.$$

Next we introduce a class of domains on which the Friedrichs operator is simpler (finite rank) and more accessible. A domain $\Omega \subset \mathbf{C}$ is called after Aharonov and Shapiro ([2]) a **quadrature domain** if there exists a distribution u of finite support in Ω, with the property that:

$$\int_\Omega f dA = u(f), \qquad f \in AL^2(\Omega).$$

The dimension of the space $L^2(\Omega) \ominus Ker(u)$ is the **order** $o(\Omega)$ of Ω. A quadrature domain of order 1 is necessarily a disk, [2]. By writing

$$u(f) = \sum_{i=1}^{n} \sum_{j=0}^{m_i} c_{ij} f^{(j)}(a_i), \qquad f \in AL^2(\Omega), \tag{3}$$

with $a_i \in \Omega, 1 \le i \le n$, and $c_{im_i} \neq 0$, we find that:

$$\mathbf{1} = \sum_{i=1}^{n} \sum_{j=0}^{m_i} \overline{c_{ij}} \frac{\partial^j k_{a_i}}{\partial \overline{a_i}^j}. \tag{4}$$

We call a_i the **nodes** of the quadrature identity (3), and c_{ij} its **weights**.

Quadrature domains are very rigid; for instance their boundary is an irreducible real algebraic curve with very special singularity points. The class of simply connected quadrature domains coincides with the conformal images of the unit disk by rational maps. For further details concerning quadrature domains see [24] and the references cited there.

2.3. **First observations.** Several simple facts concerning the Friedrichs operator are already within reach. The notations are those introduced before.

Lemma 2.1. *Asume the domain Ω is bounded and let $f \in L^2(\Omega)$ be a solution of the equation $Sf = f$. Then f is a constant function.*

Proof. Indeed, from $Sf = f$ one obtains $\|Ff\| = \|f\|$, hence $Cf = PCf$. Thus Cf is both an anti-analytic and an analytic function, so f is necessarily a constant. □

Lemma 2.2. *Let $\phi \in H^\infty(\Omega)$. Then:*

$$FT_\phi = {T_\phi}^* F. \tag{5}$$

Proof. A pair of functions $f, g \in AL^2(\Omega)$ satisfies:

$$\langle f, {T_\phi}^* Fg \rangle = \langle \phi f, PCg \rangle = \langle \phi f, Cg \rangle =$$
$$\langle f, C(\phi g) \rangle = \langle f, PC(\phi g) \rangle = \langle f, F(T_\phi g) \rangle.$$

□

Since $F\mathbf{1} = \mathbf{1}$, the next result follows.

Corollary 2.3. *For every function $\phi \in H^\infty(\Omega)$ one has:*

$$F\phi = {T_\phi}^* \mathbf{1}. \tag{6}$$

In particular, if $AL^\infty(\Omega)$ is dense in the Bergman space, then F is the unique anti-linear operator on $L^2(\Omega)$, satisfying relation (6).

As an application we derive the following known theorem, see [24] Theorem 8.4.

Theorem 2.4. (H. S. Shapiro) *Let Ω be a domain with boundary consisting of finitely many continua. Then Ω is a quadrature domain if and only if $rank(F) < \infty$.*
In that case $o(\Omega) = rank(F)$.

Proof. The regularity of $\partial\Omega$ in the statement implies that $AL^\infty(\Omega)$ is dense in $AL^2(\Omega)$ and moreover, there are no bounded point evaluations for $AL^2(\Omega)$ lying on $\partial\Omega$, see for details [25].

Let Ω be a quadrature domain, and assume for clarity that its nodes a_i are simple. Let $\phi \in H^\infty(\Omega)$. According to the identity $T^*_\phi k_{a_i} = \overline{\phi(a_i)}$ and formula (4), $F\phi$ belongs to the linear span of $k_{a_i}, 1 \le i \le o(\Omega)$. The case of multiple nodes is a simple generalization of this observation.

Conversely, if $rank(F)$ is finite and N is large enough, then the vectors $(T^*_z)^k\mathbf{1}$, $1 \le k \le N$, are linearly dependent. Consequently there exists a polynomial $p(z)$ of minimal degree, satisfying $T^*_p\mathbf{1} = 0$. Since there are no bounded point evaluations in $\partial\Omega$, the zeroes a_i of p lie in Ω. Then formula (4) follows. □

Actually, using more recent investigations concerning finite codimensional invariant subspaces of the Bergman space, cf. [3], we can complement the latter theorem by the following observation.

Theorem 2.5. *Let Ω be a bounded domain, no boundary component of which is a point. The following are equivalent:*

a). $im(S)$ has finite dimension k;
b). $ker(S)$ has finite codimension k;
c). $ker(F)$ has finite codimension k;
d). Ω is a quadrature domain of order k.

Proof. The equivalence between a) and b) follows from the fact that $S^* = S$. We also know that $ker(F) = ker(S)$ and that d) $\Rightarrow$ c). It remains to prove that c) implies d).

Assume that $ker(F)$ has finite codimension k. Since $ker(F)$ is invariant under multiplication by $H^\infty(\Omega)$, the Axler-Bourdon theorem [3] shows that there exists a polynomial $p(z)$, having k zeroes in Ω, with the property that $ker(F) = \{pf;\ f \in AL^2(\Omega)\}$. Thus, all elements g in $AL^2(\Omega)$ vanishing with the corresponding multiplicity on the zeroes a_i of p, satisfy $\int_\Omega g dA = 0$. Therefore the vector $\mathbf{1}$ is a linear combination of the evaluation functions k_{a_i} and possibly some of their derivatives. This implies that Ω is a quadrature domain. □

In particular, $rank(F) = 1$ if and only if Ω is a disk.

3. Planar elasticity and the Friedrichs inequality

In this section we briefly recall and comment on a part of the results due to Friedrichs [7]. A bounded planar domain Ω will be regarded as a thin, deformable plate. We adopt the hypothesis that both strain and stress tensors of Ω depend only on the two coordinates in the plane of Ω. Two fundamental questions in elasticity theory ask to find the (equilibrum) distribution of stress through Ω, in the case when the stress, respectively the strain, is known on $\partial\Omega$. A long time ago it was realized (it seems by J. C. Maxwell, [14]) that these questions can be interpreted as boundary value problems for the bi-Laplacian, hence the natural connection with complex functions in Ω. Along these lines, the two problem were satisfactorily solved at the beginning of this century by Kolosov and Muskhelishvili, see [14] for details. Their method, which is considered the standard one, consists in transforming the problems into singular integral equations (with Cauchy type kernels) and then to use the Fredholm alternative for describing the set of solutions.

Less known is the approach proposed by Friedrichs [7]. By using variational methods for the energy of the plate, he has reduced the two questions to integral equations with operators involving only the area measure on Ω. This seems to be the first occurrence and application of the Bergman space to mathematical physics. We sketch below Friedrichs' equations and a way of representing their solutions. For complete details the reader can consult [16] besides Friedrichs' paper.

Specifically, the two basic problems of planar elasticity correspond to finding elements $u, v \in AL^2(\Omega)$ such that:

$$(I+F)u = f, \tag{7}$$

respectively

$$(\kappa - F)v = g, \tag{8}$$

where $f, g \in L^2(\Omega)$ are given and $\kappa > 1$ is a constant depending on the material. The next result is the key to analyzing these equations.

Theorem 3.1. (Friedrichs) *Assume that the bounded domain Ω has piece-wise smooth boundary, with finitely many corners of interior angles $\alpha_k, 0 < \alpha_k \leq 2\pi, 1 \leq k \leq n$. Put $\alpha_0 = \pi$ for the generic smooth point of $\partial\Omega$.*

Then $\sigma_{ess}(S) \supset \{|\frac{\sin \alpha_k}{\alpha_k}|^2; 0 \leq k \leq n\}$.

Let us remark that, only external cusps are excluded in the statement from the singularities of the boundary. As we shall see later in section 6, the inclusion in Friedrichs' theorem may be strict.

To solve equation (8) it is enough to remark that

$$\kappa^2 - S = \kappa^2 - F^2 = (\kappa - F)(\kappa + F)$$

is an invertible operator ($S \leq I$). Thus $v = (\kappa - F)^{-1}g = (\kappa^2 - S)^{-1}(\kappa + F)g$ is the unique solution of (8). In order to solve equation (7) we either decompose f into $f = \langle f, \mathbf{1}\rangle\mathbf{1} + (f - \langle f, \mathbf{1}\rangle\mathbf{1})$ and correspondingly split (7) along $\mathbf{C1}$ and $\mathbf{1}^\perp$, or apply $(I - F)$ to (7). Either way, according to Lemma 2.1, the equation on the orthogonal complement of $\mathbf{1}$ has a unique solution. The equation along $\mathbf{1}$ reduces to $(I+F)\sigma = \lambda$, with complex numbers σ, λ. This is possible if and only if $\lambda \in \mathbf{R}$, and in that case $\Re\sigma = \lambda/2$. All in all, equation (7) is solvable if and only if $\langle f, \mathbf{1}\rangle \in \mathbf{R}$. In that case the solution is:

$$u = \langle f, \mathbf{1}\rangle/2 + iy + (I+F)^{-1}(f - \langle f, \mathbf{1}\rangle\mathbf{1}),$$

with an arbitrary parameter $y \in \mathbf{R}$.

As a function theoretic application of Theorem 3.1, Friedrichs has established the following inequality, which actually gives the title of his paper.

Corollary 3.2. *Let Ω be a domain satisfying the conditions in Theorem 3.1. Then there exists a positive constant $c = c(\Omega) < 1$, with the property:*

$$|\int_\Omega f^2 dA| \leq c \int_\Omega |f|^2 dA, \qquad f \in AL_0^2(\Omega). \tag{9}$$

We call (9) the **Friedrichs inequality** of the domain Ω. A different proof and a generalization to $L^1(\Omega)$ of Friedrichs' inequality was given in [20]. Another generalization of Friedrichs' inequality to $AL^p(\Omega), 1 < p < \infty$, where Ω is a domain satisfying the interior cone condition at each boundary point, appears in [22]. The main idea there is to use the theory of strongly elliptic systems of differential operators in order to establish that the algebraic sum $AL^p(\Omega) + CAL^p(\Omega)$ is closed in $L^p(\Omega), 1 < p < \infty$, whenever Ω satisfies the interior cone condition. As a matter of fact Friedrichs' theorem above and Corollary 3.2 hold for such domains.

Proof. (of Corollary 3.2). Since the self-adjoint operator $I - S$ is Fredholm and $\mathbf{1}$ spans its kernel, we obtain:

$$c^2 = \sup\{\langle Sf, f\rangle | \langle f, \mathbf{1}\rangle = 0, \|f\| \leq 1\} < 1.$$

Therefore:

$$|\int_\Omega f^2 dA| = |\langle f, Ff\rangle| \leq \|f\|\|Ff\| \leq \|f\|\langle Sf, f\rangle^{1/2} \leq c\|f\|^2.$$

□

Actually the above argument can be reversed and it yields.

Proposition 3.3. *A bounded planar domain Ω satisfies Friedrichs' inequality if and only if* $\sup\{\lambda; \lambda \in \sigma_{ess}(S)\} < 1$.

Proof. Let $\gamma = \sup\{\lambda | \lambda \in \sigma_{ess}(S)\}$ and assume $\gamma < 1$. We know that 1 is the largest eigenvalue of S, of multiplicity one, cf. Lemma 2.1. Hence the interval $(\gamma, 1)$ contains only finitely many elements in $\sigma(S)$, all lying in the point spectrum of S. Let c^2 be the largest element, if the latter set is non-empty, or let $c^2 \in (\gamma, 1)$ be arbitrary, if the set is empty. Then

$$\langle Sf, f\rangle \leq c^2\|f\|^2, \qquad f \in AL^2_0(\Omega).$$

Thus, Friedrichs' inequality holds for Ω.

Conversely, if Friedrichs' inequality is satisfied, then obviously $\gamma < 1$. □

4. Spectral theory of the Friedrichs operator

In this section we investigate the kernel and the range of the Friedrichs operator, its essential spectrum and other spectral invariants. We will prove that, in general, the spectrum of F does not determine the underlying domain, even up to affine motions. Simple arguments show that translations and homotheties of Ω in the complex plane produce unitarily equivalent Friedrichs operators.

4.1. Fredholm theory.

Below, an $\mathbf{R}$-linear operator T is called **Fredholm** if both $ker(T)$ and $coker(T)$ are finite dimensional (the latter condition implies that $im(T)$ is closed). Most of the Fredholm theory carries to $\mathbf{R}$-linear operators, cf. [11].

Theorem 4.1. *Let Ω be a planar domain and let F be its Friedrichs operator. Then $ker(F)$ is 0- or ∞-dimensional and F is not Fredholm. In particular, $0 \in \sigma_{ess}(S)$.*

Proof. According to relation (5), $ker(F)$ is $H^\infty(\Omega)$-invariant. Hence its dimension is either zero, or infinity.

Assume by contradiction that F is Fredholm, that is $ker(F) = 0$ and $coker(F)$ has finite dimension. Pick distinct points $w_j \in \Omega, 1 \leq j \leq N$. If N is big enough, then there exists a non-trivial linear combination $v = \sum_{j=1}^N \alpha_j k_{w_j}$ belonging to $im(F)$, the range of F. Let $p(z)$ be a non-trivial polynomial which vanishes at the points $w_j, 1 \leq j \leq N$.

It is easy to verify that the evaluation vector k_w satisfies $T_\phi{}^* k_w = \overline{\phi(w)} k_w$ for every function $\phi \in H^\infty(\Omega)$. Thus $T_p^* v = 0$. But relation (5) implies $F(pg) = T_p^* v = 0$, where $Fg = v$, a contradiction to the fact that F is injective.

Since $S = F^2$, the operator S cannot be Fredholm. □

For a quadrature domain, the Friedrichs operator has infinite dimensional kernel and finite dimensional range. There are many examples of domains for which the Friedrichs operator is injective. For instance the ellipse or an annulus, see section 6 . The preceding proof has the following corollary which may be of independent interest.

Corollary 4.2. *For every domain Ω possessing injective Friedrichs operator, no linear combination of the evaluation functions $k_w, w \in \Omega$, lies in $im(F)$.*

Next, keeping in mind the example of quadrature domains, we focus on the behaviour of all functions belonging to $ker(F)$. We say that the domain Ω admits a **generalized quadrature** supported by the closed set $K \subset \overline{\Omega}$, if there exists a complex measure μ on K, so that the total variation $|\mu|$ of μ does not charge $\partial\Omega$, and with the property that:

$$\int_\Omega f dA = \int_K f d\mu, \qquad f \in O(\overline{\Omega}).$$

In that case $\partial\Omega \setminus K$ is a real analytic (possibly singular) curve.

Instead of the test space $O(\overline{\Omega})$ above one can take $AL^1(\Omega)$, by assuming in this case that

$$\int_K \frac{|d\mu|(z)}{dist(z, \partial\Omega)} < \infty.$$

If the domain Ω allows a generalized quadrature in the above sense, then $\partial\Omega \setminus K$ is a real analytic curve possibly with inward pointing cusps, cf. [24] Chapter 5.

Again an example is the ellipse, in which case K is the segment joining the foci.

Proposition 4.3. *Let Ω be a domain with the property that $O(\overline{\Omega})$ is dense in $AL^2(\Omega)$. Suppose that Ω satisfies a generalized quadrature formula supported by a compact set K, so that ∂K does not separate the plane.*

Then either $ker(F) = 0$, or the set K can be chosen discrete and contained in the zero set of a non-trivial function in $AL^2(\Omega)$. In that case,

$$ker(F) = \{f \in AL^2(\Omega); f|_K = 0\}.$$

Proof. According to Mergelyan's theorem, the functions $g \in AL^2(\Omega)$ are dense in $C(K)$, the algebra of continuous functions on K. Let $f \in ker(F)$. Then

$$0 = \langle f, Fg \rangle = \int_\Omega (fg) dA = \int_K (fg) d\mu.$$

Consequently $fd\mu = 0$ and we can choose K to be contained in the zero set of f. With this choice, any function $h \in AL^2(\Omega)$ vanishing on K is necessarily annihilated by the Friedrichs operator. □

The argument in the preceding proof can be extended to classes of sets K whose boundary may separate the plane; for instance by approximating the continuous functions on K with rational functions with poles off K. We do not expand here these details.

We do not know other domains than quadrature domains for which the Friedrichs operator has closed range.

4.2. **The kernel of the Friedrichs operator.** This part of the paper deals with functions which are annihilated by the Friedrichs operator. The following example shows that, in the absence of a nice generalized quadrature formula, Proposition 4.3 may fail.

Theorem 4.4. *There exists a Jordan domain with C^∞ boundary, with the property that its Friedrichs operator annihilates a nowhere vanishing function.*

In other terms we will construct a domain Ω and a function $g \in AL^2(\Omega)$, with the property that $g(z) \neq 0, z \in \Omega$, and

$$\int_\Omega fg dA = 0, \qquad f \in AL^2(\Omega). \tag{10}$$

The proof is based on the following result, of independent interest. Below we denote by $H^2(\mathbf{D})$ the Hardy space of the unit disk.

Proposition 4.5. *There exists a singular inner function J in the unit disk, with the property that $H^2(\mathbf{D}) \ominus JH^2(\Omega)$ contains a univalent function $u \in H^2(\mathbf{D}) \cap C^\infty(\partial\mathbf{D})$.*

Proof. *(of Theorem 4.4).* Assume the existence of the univalent function u in Proposition 4.5. A power series expansion argument shows that, for any pair $f, g \in O(\overline{\mathbf{D}})$, relation:

$$\langle f', g\rangle_{2,\mathbf{D}} = \langle f, zg\rangle_{2,\partial\mathbf{D}} \tag{11}$$

holds, where the right hand side is the scalar product in $H^2(\mathbf{D})$. Therefore,

$$\langle u', Jf\rangle_{2,\mathbf{D}} = \langle u, zJf\rangle_{2,\partial\mathbf{D}}.$$

By changing variables $w = u(z)$, and letting $z = v(w)$, we have a biholomorphic map $u : \mathbf{D} \longrightarrow \Omega$, with inverse v defined on the bounded domain Ω. According to the previous computations, for all $f \in O(\overline{\mathbf{D}})$, we obtain:

$$0 = \langle Jf, u'\rangle_{2,\mathbf{D}} = \int_{\mathbf{D}} \overline{u'(z)} J(z) f(z) dA(z) =$$

$$\int_\Omega \overline{u'(v(w))} J(v(w)) f(v(w)) v'(w) \overline{v'(w)} dA(w) =$$

$$\int_\Omega J(v(w)) f(v(w)) v'(w) dA(w).$$

Thus, denoting $g(w) = J(v(w))v'(w)$, we see that:

$$\int_\Omega |g|^2 dA \leq \int_\Omega |v'(w)|^2 dA(w) = |\mathbf{D}| = \pi,$$

so $g \in AL^2(\Omega)$ and g does not vanish on Ω. Since the functions $f(v(w)), f \in O(\overline{\mathbf{D}})$ are dense in $AL^2(\Omega)$, relation (10) follows. □

Proof. *(of Proposition 4.5).* Let I be a singular inner function with associated measure $d\mu$, singular with respect to dt:

$$I(z) = exp[\int_{-\pi}^{\pi} \frac{z + e^{it}}{z - e^{it}} d\mu(t)].$$

If the support of $d\mu$ is small enough, for instance a point, there exists an outer function O, vanishing on that set, so that $\overline{O}I \in C^r(\partial\mathbf{D})$, for a prescribed $r > 1$, including the possibility $r = \infty$.

Let $f(z) = \overline{zO(z)}I(z)$ and denote by $\Pi : L^2(\partial\mathbf{D}) \longrightarrow H^2(\partial\mathbf{D})$ be the Szegö projection. Then Πf is a non-trivial element of $(IH^2(\partial\mathbf{D}))^\perp$. Indeed, for all $b \in H^\infty(\mathbf{D})$ the following identities, in the scalar product of H^2:

$$\langle \Pi f, Ib\rangle = \langle \Pi(\overline{zO}I), Ib\rangle = \langle \overline{zO}I, Ib\rangle =$$

$$\langle I, zOIb\rangle = \langle 1, zOb\rangle = 0,$$

hold.

Moreover, $\Pi f \neq 0$, because otherwise, $0 = \langle \overline{zO}I, b\rangle = \langle I, zOb\rangle$. But O is outer, so $OH^\infty(\partial\mathbf{D})$ is dense in the Hardy space. This would imply that f is constant, a contradiction.

Next we will produce from f a univalent function u as in the statement. If $u(z) = z + \sum_{j=2}^\infty c_j z^j$, we use

$$\sum_{j=2}^\infty j|c_j| < 1 \tag{12}$$

as an injectivity criterion for u, see [26] p.212, Problem 8.

Since all backward shifts preserve $(IH^2)^\perp$, we can assume that:

$$f(z) = b_0 + z + \sum_{j=2}^\infty b_j z^j.$$

We will use an averaging process, in order to annihilate a part of the Taylor coefficients of f. Pick a positive integer k, let $\omega = e^{2\pi i/k}$ and write $h(z) = (f(z) - b_0)/z$. Then:

$$\frac{h(z) + h(\omega z) + \ldots + h(\omega^{k-1}z)}{k} = \sum_{j=0}^\infty b_{kj+1} z^{kj}.$$

Thus,

$$\frac{f(z) - b_0}{z} + \frac{f(\omega z) - b_0}{\omega z} + \ldots + \frac{f(\omega^{k-1}z) - b_0}{\omega^{k-1}z} = k\sum_{j=0}^\infty b_{kj+1} z^{kj},$$

or,

$$u(z) = \frac{f(z) + \overline{\omega} f(\omega z) + \ldots + \overline{\omega^{k-a}} f(\omega^{k-a}z)}{k} = \sum_{j=0}^\infty b_{kj+1} z^{kj+1}.$$

Now the function u is no more orthogonal to IH^2. Instead, it is orthogonal to JH^2, where

$$J(z) = I(z)I(\omega z)\ldots I(\omega^{k-1}z).$$

By picking k large enough we can assure that condition (12) is met. Moreover, u inherits from f the regularity to the boundary. □

The above proof allows to construct a univalent function u, orthogonal to JH^2, so that $u \in C^\infty(\overline{\mathbf{D}})$. Moreover, u can be a small perturbation of the identity map, so the domain $\Omega = u(\mathbf{D})$ can be constructed to be arbitrarily close to the unit disk. A morphological feature of the function $g \in ker(F_\Omega)$ is that it tends to zero rather fast along at least one path in Ω leading to a boundary point.

The next result shows that the phenomenon described in Theorem 4.4 cannot hold for domains with smooth real analytic boundary.

Theorem 4.6. *Let Ω be a domain with generalized quadrature formula given by a measure with compact support in Ω. If $kerF$ contains a non-trivial function f, then f must have a zero in Ω.*

Proof. Let E be a compact subset of Ω containing the support of a generalized quadrature positive measure μ. Suppose that $F(f) = 0$ and that f does not vanish in Ω. By Runge's theorem there exists a sequence R_n of rational functions without poles in $\overline{\Omega}$ converging uniformly to $1/f$ on E. Thus $R_n f$ converges uniformly to $\mathbf{1}$ on E, so:

$$\lim_{n\to\infty} \int_\Omega R_n f dA = \lim_{n\to\infty} \int_E R_n f d\mu = \int_E d\mu = Area(\Omega).$$

But all the integrals $\int_\Omega R_n f dA$ are zero by assumption, a contradiction. □

Since all domains with smooth real analytic boundary possess such generalized quadrature formulas, see for instance [9], the next corollary follows.

Corollary 4.7. *If Ω is a bounded domain with smooth real analytic boundary, then any non-trivial function in $ker(F)$ vanishes somewhere in Ω.*

Next we relate the kernel of the Friedrichs operator to biholomorphic functions. Recall that a **biholomorphic function** u is a solution of the equation $\overline{\partial}^2 u = 0$. Such functions are always representable as $u(z) = f(z) + \overline{z}g(z)$, with f, g analytic. The maximun priciple does not hold for this class of functions, for instance there are domains Ω for with the property that there exist biholomorphic functions in Ω, which are identically zero on the boundary, see [6].

Theorem 4.8. *Let Ω be a bounded domain with smooth boundary. Then there exists a biholomorphic function u in Ω, continuous on $\overline{\Omega}$ and such that $u|_{\partial\Omega} = 0$ if and only if the kernel of the Friedrichs operator is non-trivial.*

Proof. Assume that $f \in ker(F)$. According to Havin's lemma [10] there exists a function $u \in W_0^{(1,2)}(\Omega)$ with the properties: $\overline{\partial} u = f$ and $\overline{\partial}^2 u = 0$, in Ω.

Conversely, assume that $v \in W^{(1,2)}(\Omega)$ satisfies $\overline{\partial}^2 v = 0$ and $v|_{\partial\Omega} = 0$. Then, again by Havin's lemma, $f = \overline{\partial} u \in AL^2(\Omega)$ and by Stokes theorem $\int_\Omega fgdA = 0$, $g \in AL^2(\Omega)$. Thus $Ff = 0$. □

4.3. **The inverse spectral problem.** A natural question is whether the spectrum of the Friedrichs operator characterizes the domain, up to the group of affine transformations of $\mathbf{C}$. Here by affine transformation we mean a map of the form $z \to az + b$ with $a, b \in \mathbf{C}$, $a \neq 0$. The next result settles in the negative this question.

Proposition 4.9. *There exists a continuous family of quadrature domains of order three, with the same Friedrichs operator (up to unitary equivalence) and such that no two domains in the family are related by an affine transformation of* $\mathbf{C}$.

Proof. We base our proof on a count of parameters. Let

$$f_{c,d}(z) = \frac{c_1}{z - d_1} + \frac{c_2}{z - d_2} + \ldots + \frac{c_n}{z - d_n},$$

be a rational function with poles outside the closed unit disk and such that it is univalent on $\mathbf{D}$. We can choose the n-tuples $c = (c_1, \ldots, c_n), d = (d_1, \ldots, d_n)$ with independent entries, lying in open balls $c \in B_r(c^0) \subset \mathbf{C}^n$ and $d \in B_r(d^0) \subset \mathbf{C}^n$. Thus the family of quadrature domains:

$$\Omega_{c,d} = f_{c,d}(\mathbf{D}),$$

depends on $4n$ real parameters. Two such domains are affinely equaivalent if and only if there is a Moebius transform M of the unit disk onto itself and complex numbers $a, b,\ (a \neq 0)$, such that:

$$f_{c,d}(z) = a f_{c',d'}(M(z)) + b.$$

Therefore, the orbits of this group action on $f_{c,d}$ have dimension at most 7.

In conclusion there is a family depending on $4n-7$ parameters, of mutually non-equivalent domains. On the other hand, the spectrum of the square of the Friedrichs operator (which has rank exactly equal to n) contains n eigenvalues (counting multiplicities), among which the highest is always 1. This leaves at most $n-1$ independent eigenvalues. The statement follows as soon as $4n-7-(n-1) = 3n-6 > 0$. The first value is $n = 3$. $\square$

The borderline case $n = 2$ will be illustrated in section 6 , where we will show that $\sigma(F) = \{1, \lambda\}$ parametrizes (through λ) a continuous family of quadrature domains of order 2.

A variety of variants of the inverse spectral problem can now be foreseen. We discuss below a situation in which the Friedrichs operator and its square (modulo unitary equivalence) determine the quadrature domain (modulo a finite set).

Let Ω be a bounded quadrature domain of order n and let $p(z)$ be the monic polynomial of degree n vanishing at the nodes of Ω. We will denote by

$$a_{kl} = \int_\Omega z^k \overline{z}^l dA(z),$$

the moments of Ω. It is proved in [17] that, knowing $p(z)$ and the partial matrix of moments $(a_{kl})_{0 \leq k \leq l \leq n-1}$, the domain Ω can be reconstructed. More specifically, the irreducible polynomial $P(z, \overline{z})$ which defines $\Omega \equiv \{z \in \mathbf{C}; P(z,\overline{z}) < 0\}$ is precisely:

$$P(z,\overline{z}) = \|p(z)\|^2 exp[\frac{-1}{\pi} \sum_{k,l=0}^{n-1} \frac{a_{kl}}{z^{k+1}\overline{z}^{l+1}}] - \ (terms\ in\ 1/z\ or\ 1/\overline{z}). \tag{13}$$

Above," $\equiv$ " means equal up to a finite set.

Let $B = T_z$ be the Bergman shift on Ω. Since Ω is a quadrature domain, the space

$$im(F) = \bigvee_{k=0}^{n-1} B^{*k}\mathbf{1} = \bigvee_{k=0}^{\infty} B^{*k}\mathbf{1}$$

has dimension $n = o(\Omega)$. The operator $S = F^2$ has the same range and it is of rank n, whence $S|_{imF}$ is injective. For a pair of elements $f, g \in im(F)$ we remark the identity:

$$\langle (S|_{imF})^{-1} F f, F g \rangle = \langle g, f \rangle. \tag{14}$$

Indeed, denoting $Ff = F^2 h$, we have

$$\langle (S|_{imF})^{-1} F^2 h, F g \rangle = \langle h, F g \rangle = \langle g, F h \rangle = \langle g, f \rangle.$$

Theorem 4.10. *Two quadrature domains Ω, Ω' with Friedrichs operators F, S, respectively F', S', coincide up to a finite set if and only if there exists a unitary operator $U : im(F) \longrightarrow im(F')$ with the properties:*

$$\left\{ \begin{array}{l} U(S|_{imF}) = (S'|_{imF'})U \\ UFq = F'q, \qquad q \in \mathbf{C}[z], \quad deg(q) \leq \max(o(\Omega), o(\Omega')). \end{array} \right. \tag{15}$$

Proof. Assume that the unitary U in the statement exists. The relation $UF = F'$ is symmetric. So, if a polynomial q of degree less or equal than $o(\Omega)$ satisfies $Fq = 0$, then $F'q = 0$, and conversely. Therefore $o(\Omega) = o(\Omega')$ and the two quadrature domains have the same nodes, counting also their multiplicities. Let $p(z)$ be the monic polynomial of degree $o(\Omega)$ vanishing at these nodes.

According to relations (14), for all $k, l < o(\Omega)$, we obtain:

$$\langle z^k, z^l \rangle_{2,\Omega'} = \langle (S'|_{imF'})^{-1} F'(z^l), F'(z^k) \rangle_{2,\Omega'} =$$

$$\langle U(S|_{imF})^{-1} F(z^l), UF(z^k) \rangle_{2,\Omega'} =$$

$$\langle (S|_{imF})^{-1} F(z^l), F(z^k) \rangle_{2,\Omega} = \langle z^k, z^l \rangle_{2,\Omega}.$$

Then we apply (13) and the proof is complete. □

Corollary 4.11. *Two quadrature domains Ω, Ω' with the same nodes coincide if and only if relation (15) holds for all polynomials q of degree $deg(q) < o(\Omega)$.*

In other terms, the above criterion says that a quadrature domain is determined up to a finite set by the pair of matrices $(S|_R, B^*|_R)$, both acting on the finite dimensional space $R = im(F)$. The quadrature identity can easily be obtained in these terms:

$$\int_\Omega f dA = \langle \mathbf{1}, f^*(B^*|_R)\mathbf{1} \rangle, \qquad f \in AL^2(\Omega),$$

where $f^*(z) = \overline{f(\overline{z})}$. Indeed, for $f \in H^\infty(\Omega)$ we have:

$$\langle \mathbf{1}, f^*(B^*|_R)\mathbf{1} \rangle = \langle \mathbf{1}, T_f^* \mathbf{1} \rangle = \langle T_f \mathbf{1}, \mathbf{1} \rangle = \langle f, \mathbf{1} \rangle.$$

Note that the quadrature nodes a_j coincide, including multiplicities, with the spectrum of $(B^*|_R)^*$. Moreover, the correlation matrix formed by the eigenvectors k_{a_j} of $B^*|_R$ is a part $(K = K(a_j, a_k))$ of the Bergman kernel. Another similar, and related, set of matricial invariants for a quadrature domains appears in [17].

5. Generalized quadratures and the restriction operator

We continue the analysis begun in Proposition 4.3, of the relation between the Friedrichs operator and generalized quadratures. By exploiting the explicit integral representations for the Friedrichs operator and the restriction operator to a quadrature space (to be defined below), we describe a possible way of computing the eigenvalues, and sometimes even the eigenfunctions, of S. The basic data needed for such computations are the Bergman kernel of the domain and the (generalized) quadrature formula. The last section of this paper contains several examples obtained from this principle.

First an observation of independent interest which might be useful in later proofs.

Lemma 5.1. *Let Ω be a domain with a generalized quadrature given by the compactly supported complex measure $d\mu$. Then $S = F^2$ is trace class and:*

$$Tr(S) = \int_{\Omega\times\Omega} K(z,w)^2 d\mu(z)d\mu(w). \tag{16}$$

Proof. The nuclearity of S is well known (see also Theorem 5.7 below). For more general results (although only in the case of simply connected domains) see [13]. By iterating formula (1) we obtain for every pair of functions $f, g \in AL^2(\Omega)$:

$$\langle Sf, g\rangle = \int_{\Omega\times\Omega} K(z,w)f(z)\overline{g(w)}dA(z)dA(w) =$$

$$\int_{\Omega\times\Omega} K(z,w)f(z)\overline{g(w)}d\mu(z)d\mu(w).$$

Let $u_j(z), 1 \le j < \infty$, be a complete system of orthonormalized eigenfunctions of S. Then

$$Tr(S) = \sum_{j=1}^{\infty}\langle Su_j, u_j\rangle =$$

$$\int_{\Omega\times\Omega} K(z,w)[\sum_{j=1}^{\infty} u_j(z)\overline{u_j(w)}]d\mu(z)d\mu(w) = \int_{\Omega\times\Omega} K(z,w)^2 d\mu(z)d\mu(w).$$

□

Recall that every bounded domain with smooth real analytic boundary fulfills the condition in Lemma 5.1. Also, Lemma 5.1 remains valid for some domains allowing quadrature with measures which are not necessarily carried by a compact subset; this time the convergence of the integral (16) must be assumed. Formula (16) remains valid, with the same proof, for distributions instead of quadrature measures (for instance in the case of quadrature domains).

In order to give an application of Lemma 5.1, we consider a quadrature domain Ω with simple nodes:

$$\int_{\Omega} f dA = \sum_{j=1}^{n} c_j f(a_j), \qquad f \in AL^2(\Omega). \tag{17}$$

Let us denote by K the positive definite matrix $[K(a_i, a_j)]_{i,j=1}^n$ and by $Q = diag(c_1, \dots, c_n)$ the positive matrix given by the quadrature weights.

Then, for $f \in AL^2(\Omega)$, we obtain:

$$(Ff)(z) = \int_{\Omega} K(z,w)\overline{f(w)}dA(w) = \sum_j K(z,a_j)\overline{c_j f(a_j)}.$$

The space $im(F)$ is generated by the linearly independent vectors $k_{a_j}, 1 \le j \le n$. With respect to this basis F is represented, as an antilinear operator, by the matrix K^tQ^*. This yields the following formula.

Lemma 5.2. *Let Ω be a quadrature domain with quadrature (17). Then*

$$det(S) = |c_1c_2\dots c_n|^2 det(K)^2. \tag{18}$$

Concerning the range of the Friedrichs operator, we note the following generalization of Corollary 4.2.

Proposition 5.3. *Let Ω be a domain with $ker(F) = 0$. Let $A \subset \Omega$ be the zero set of a non-trivial $H^\infty(\Omega)$-function and let H_A be the closed linear subspace of $AL^2(\Omega)$ spanned by $k_w,\ w \in A$.*

Then $im(F) \cap H_A = 0$.

Proof. The proof repeats that of Theorem 4.1. Namely if A is the zero set of $\phi \in H^\infty(\Omega)$, then $0 = T_\phi^* F g = F(T_\phi)g$ for a non-trivial vector $Fg \in H_A \cap im(F)$. This implies, via the uniqueness principle, that $Fg = 0$. □

One consequence is in order at this point. Assume that F is injective and compact, and let $f_j, 0 \le j < \infty$, be an orthonormal system of eigenfunctions for S:

$$\int_\Omega f_i f_j dA = \lambda_i \int_\Omega f_i \overline{f_j} dA, \qquad i, j \ge 0. \tag{19}$$

We can arrange the eigenvalues $\lambda_j > 0$ in decreasing order ($\lambda_0 = 1, f_0 =$ *const*). In other terms:

$$F(f_j) = \lambda_j(f_j), \quad S(f_j) = \lambda_j^2 f_j, \qquad j \ge 0.$$

Fix a point $a \in \Omega$. Then, by expanding the evaluation function k_a in a series in f_j, and applying Proposition 5.3, or even Corollary 4.2, we see that the series cannot terminate. Thus we have proved the next corollary.

Corollary 5.4. *Let Ω be a domain with compact, injective Friedrichs operator. Then for every point of Ω there are infinitely many Friedrichs eigenfunctions non-vanishing at that point.*

The spaces H_A, with A a subset of Ω, appear in the reconstruction of an analytic function from its values on A; more specifically they appear in the extremal problem of finding the function of least L^2-norm when its values on A are prescribed, cf. [23]. We elaborate below such an aspect.

Let Ω be a (not necessarily bounded) domain with Bergman kernel $K(z, w)$. The Friedrichs operator F is given by the integral formula (1), while its square S acts as:

$$Sf(z) = \int_\Omega L(z, w) f(w) dA(w), \qquad f \in AL^2(\Omega), \tag{20}$$

where $L(z, w) = \int_\Omega K(z, \zeta) K(w, \zeta) dA(\zeta)$. Note an unusual fact, namely that the kernel $L(z, w)$ is symmetric in z, w, and not Hermitean-symmetric as would necessarily be the case for the kernel representing a self-adjoint operator on the whole space $L^2(\Omega)$.

Consider a positive measure μ on Ω, whose support E may intersect $\partial\Omega$, but such that $\mu(\partial\Omega) = 0$. We suppose that $E \cap \Omega$ is a set of uniqueness for $AL^2(\Omega)$, and also that there exists a positive constant $C = C(\Omega, \mu)$ with the property:

$$\int_E |f|^2 d\mu \le C \int_\Omega |f|^2 dA, \qquad f \in AL^2(\Omega). \tag{21}$$

In this case the restriction operator $R(f) = f|_E$ is well defined and bounded as a linear operator $R : AL^2(\Omega) \longrightarrow L^2(E, \mu)$. By our assumption, R is injective, and we will also assume that R has dense range. Its adjoint has the explicit representation:

$$R^*\psi(z) = \int_E K(z, w)\psi(w) d\mu(w), \qquad z \in \Omega,\ \psi \in L^2(E, \mu). \tag{22}$$

We study below an important particular case of this setting. Let us denote by $HL^1(\Omega), SL^1(\Omega)$ the spaces of integrable harmonic, respectively subharmonic, functions in Ω. Suppose that Ω is symmetric with respect to the real axis and admits a quadrature identity of the type:

$$\int_\Omega u dA = \int u d\mu, \qquad u \in HL^1(\Omega), \tag{23}$$

where μ is a positive measure on $\mathbf{R} \cap \Omega$. Due to the symmetry assumption and a theorem of Sakai, [19] §9, we have in these conditions the much stronger estimate:

$$\int s d\mu \leq \int_\Omega s dA, \qquad s \in SL^1(\Omega). \tag{24}$$

In particular relation (21) is satisfied with the constant $C = 1$.

Let $f^*(z) = \overline{f(\overline{z})}$, $z \in \Omega$ be the involution induced by the symmetry with respect to the axis $\mathbf{R}$.

Theorem 5.5. *Let Ω be a symmetric domain with respect to the real axis, which satisfies a generalized quadrature (23) and let*

$$R : AL^2(\Omega) \longrightarrow L^2(\mathbf{R}, \mu)$$

be the corresponding restriction operator. Then $S = (R^*R)^2$ and $S(f^*) = (Sf)^*$, $f \in AL^2(\Omega)$.

Proof. By the symmetry assumption and the transformation law of the Bergman kernel under this symmetry, we have $K(x, y) = K(y, x) \in \mathbf{R}$ for $x, y \in \mathbf{R} \cap \Omega$.

For a function $f \in AL^2(\Omega)$ we have:

$$R^*Rf(z) = \int K(z, x)f(x)d\mu(x),$$

so

$$(R^*R)^2 f(z) = \int K(z, y)[\int K(y, x)f(x)d\mu(x)]d\mu(y) =$$
$$\int f(x)[\int K(z, y)K(y, x)d\mu(y)]d\mu(x) =$$
$$\int f(x)[\int K(z, y)K(x, y)d\mu(y)]d\mu(x).$$

By using twice the quadrature identity we obtain:

$$(R^*R)^2 f(z) = \int f(x)[\int K(z, w)K(x, w)dA(w)]d\mu(x) =$$
$$\int f(x)L(z, x)d\mu(x) = \int f(w)L(z, w)dA(w) = Sf(z).$$

From the same computations we infer $S(f^*) = (Sf)^*$. □

The commutativity between S and the involution " $*$ " in Theorem 5.5 implies that all spectral subspaces of S are invariant under $*$, therefore they are generated by invariant functions: $f^* = f$. In particular, it will be enough in such situations to solve the eigenfunction equation $Sf = \lambda f$ for f satsisfying $f^* = f$, or equivalently, $f(x) \in \mathbf{R}$, $x \in \mathbf{R} \cap \Omega$.

Corollary 5.6. *In the conditions of Theorem 5.5, the operator $\sqrt{S}$ is unitarily equivalent to the integral operator $RR^* : H \longrightarrow H$:*

$$(RR^*f)(x) = \int_{\mathbf{R}\cap\Omega} K(x, y)f(y)d\mu(y), \qquad f \in H,$$

where H is the closure of the range of R in $L^2(\mathbf{R}, \mu)$.

Proof. For the proof is suffices to use relation (22) and recall that the operators R^*R and $RR^*|_H$ are unitarily equivalent, for a proof see for instance [22]. □

In most applications $H = L^2(\mathbf{R}, \mu)$.

The next result was stated without proof in [24] (see the remarks following [24], Theorem 8.5).

Theorem 5.7. *Let Ω be a bounded domain which admits a generalized quadrature formula with the associated measure compactly supported by Ω. Then S is compact and its sequence of eigenvalues λ_n (arranged in decreasing order) satisfies:*

$$\limsup_{n\to\infty} \lambda_n^{1/n} < 1. \tag{25}$$

Proof. Assume that the measure μ is supported by the compact set $E \subset \Omega$ and it satisfies:

$$\int_\Omega f dA = \int_E f d\mu, \qquad f \in AL^2(\Omega).$$

Then it is known that $\partial\Omega$ consists of finitely many analytic arcs. Since $Ff(z) = \int_\Omega K(z,w)\overline{f(w)}dA(w) = \overline{\int_E K(w,z)f(w)d\mu(w)}$, there exists a positive constant $C(= \max_{z\in E} K(z,z))$ such that:

$$\|Ff\| \le C \max_{z\in E} |f(z)|, \qquad f \in AL^2(\Omega).$$

Consequently,

$$\|Sf\| = \|F(Ff)\| \le C\|Ff\|_{\infty,E} \le C^2\|Ff\|_{2,\Omega} \le C^3\|f\|_{\infty,E}.$$

Thus, there exists a positive constant B with the property:

$$\|Sf\|_{2,\Omega} \le B\|f\|_{\infty,E}, \qquad f \in AL^2(\Omega). \tag{26}$$

This proves the compactness, even nuclearity, of S.

In order to estimate the decay of the eigenvalues of S we use the Fischer-Courant mimimax criterion: $\lambda_0 = \max\{\langle Sf, f\rangle;\ \|f\| = 1\} = 1$, and for $n \ge 1$:

$$\lambda_n = \min_{X_n} \max\{\langle Sf, f\rangle,\ f \in X_n,\ \|f\| = 1\}, \tag{27}$$

where X_n ranges over all subspaces of $AL^2(\Omega)$ having codimension n.

Let now G be an open subset of Ω such that $E \subset G$ and $\overline{G} \subset \Omega$. Then we can find a function $\phi \in H^\infty(\Omega)$ such that:

$$m_1 = \sup_{z\in E} |\phi(z)| < \inf_{z\in\Omega\setminus\overline{G}} |\phi(z)| = m_2. \tag{28}$$

For example, if ϕ is a nonconstant function in $H^\infty(\Omega)$ satisfying $\lim_{z\to\zeta} |\phi(z)| = 1$ for all $\zeta \in \partial\Omega$, then one can define G so that relation (28) holds.

From (28) it is clear that the set of zeroes of ϕ is non-empty and finite; let r denote their number, counting multiplicities.

We shall first derive an upper bound for λ_n when n is divisible by r, say $n = kr$. We will need the following lemma, whose proof is an application of the maximum principle, and left to the reader.

Lemma 5.8. *If K is a compact subset of Ω, then there is a positive constant $C = C(\Omega, K)$ such that:*

$$\int_\Omega |g|^2 dA \le C \int_{\Omega\setminus K} |g|^2 dA, \qquad g \in AL^2(\Omega).$$

Let Y_n denote the subspace of $AL^2(\Omega)$ consisting of all functions $f = \phi^k g$, with $g \in AL^2(\Omega)$ where ϕ satisfies (28). This space has codimension $n = kr$, so by (27):

$$\lambda_n \leq \max\{\langle Sf, f\rangle; f \in Y_n,\ \|f\| = 1\},$$

and, using (26):

$$\lambda_n \leq B \sup\{\|f\|_{\infty,E};\ f \in Y_n,\ \|f\| = 1\}.$$

Let $f = \phi^k g$ be an element of Y_n of norm 1. According to (28) we have:

$$1 = \int_\Omega |f|^2 dA \geq \int_{\Omega\setminus\overline{G}} |f|^2 dA \geq m_2^{2k} \int_{\Omega\setminus\overline{G}} |g|^2 dA.$$

Hence

$$\int_{\Omega\setminus\overline{G}} |g|^2 dA \leq (m_2)^{-2k},$$

so by the lemma, $\|g\|^2 \leq C(m_2)^{-2k}$.

From here on, we let C_j denote constants that depend only on the geometric configuration Ω, E, G. Then

$$|g(z)|^2 \leq C_1(m_2)^{-2k}, \qquad z \in E,$$

so

$$\|f\|_{\infty,E} \leq C_2(m_1)^k(m_2)^{-k} = C_2\rho^k = C_2\rho_0^n,$$

where $\rho = m_1/m_2 < 1$ and $\rho_0 = \rho^{1/r}$.

For an arbitrary n, let k be the greatest integer less than or equal to n/r. Then

$$\lambda_n \leq \lambda_{kr} \leq C_2\rho_0^{kr} \leq C_2\rho_0^{n-r} = (C_2\rho_0^{-r})\rho_0^n.$$

In conclusion,

$$\limsup_{n\to\infty} \lambda_n^{1/n} \leq \rho_0$$

and the theorem is proved. □

6. Examples

This section contains examples of domains on which the Friedrichs operator is relatively simple or at least its spectrum is computable. Due to space constraints, part of the routine computations are omitted.

6.1. **Quadrature domains of order two.** For such domains the spectrum of the Friedrichs operator is computable in terms of the Bergman kernel. For instance, let Ω be a bounded quadrature domain of order two, with a double point at $z = 0$:

$$\int_\Omega f dA = c_0 f(0) + c_1 f'(0), \qquad f \in AL^2(\Omega).$$

Then $\mathbf{1}$ and $u = \frac{\partial K(z,w)}{\overline{w}}|_{w=0}$ generate $im(F)$. Since these vectors are orthogonal, the second eigenvalue λ of F is given by the formula $Fu = \lambda u$. But

$$Fu(z) = \int_\Omega K(z,w)\overline{u(w)}dA(w) =$$

$$\overline{c_0}K(z,0)\overline{u(0)} + \overline{c_1}\frac{\partial K(z,w)}{\overline{w}}|_{w=0}\overline{u(0)} + \overline{c_1}K(z,0)\overline{u'(0)} = \overline{c_1}\overline{u(0)}u(z).$$

Thus

$$\lambda = \overline{c_1}\frac{\partial K(0,w)}{\overline{w}}|_{w=0}.$$

By choosing $\Omega_a = \{z^2 + az; |z| < 1\}$, with the real parameter $a \geq 2$, a short computation using the change of variable formula for the Bergman kernel yields:

$$\lambda(\Omega_a) = \frac{4}{a^4}.$$

Thus, in the family of quadrature domains $(\Omega_a)_{a\geq 2}$, the spectrum of their Friedrichs operator distinguishes each pair of them.

Similar computations can be carried out in the case of two distinct nodes, say $\pm a$. The eigenvectors are then $\mathbf{1}$ and $k_a - k_{-a}$.

6.2. **An annulus.** Let $A_t = \{z;\ e^{-t/2} < |z| < e^{t/2}\}$ be an annulus with a fixed parameter $t > 0$. It is easy to verify that, for conveniently chosen constants $a_n > 0$, the system of functions $f_0 = \mathbf{1},\ f_{\pm n} = z^n \pm a_n z^{-n},\ n > 0$, is a doubly orthogonal basis of $AL^2(A_\rho)$:

$$\sigma_k \langle f_k, f_l\rangle = \langle f_k, F f_l\rangle = \int_{A_\rho} f_k f_l dA, \quad k, l \in \mathbf{Z}.$$

The eigenvalues of $\sqrt{S}$ will then be $\lambda_k = |\sigma_k|,\ k \in \mathbf{Z}$.

We indicate below how to compute these numbers. First, the orthogonality condition:

$$0 = \langle f_n, f_{-n}\rangle = 2\pi \int_1^\rho (|z|^{2n} - a_n^2 |z|^{-2n})\rho d\rho,$$

yields:

$$a_n^2 = \begin{cases} \frac{\sinh 2t}{2t}, & n = 1, \\[2ex] \frac{\sinh(n+1)t}{n+1} \frac{n-1}{\sinh(n-1)t}, & n > 1. \end{cases}$$

Consequently we find the eigenvalues of $\sqrt{S}$:

$$\lambda_n = \lambda_{-n} = \frac{a_n \sinh t}{\frac{\sinh(n+1)t}{n+1}}, \qquad n > 0.$$

In order to unify the formulas, we put by convention:

$$t = \frac{\sinh nt}{n}|_{n=0}.$$

Therefore, for all values $n \in \mathbf{Z}$ one obtains:

$$\lambda_n = \sqrt{\frac{(n+1)\sinh t}{\sinh(n+1)t}} \sqrt{\frac{(n-1)\sinh t}{\sinh(n-1)t}}. \tag{29}$$

Note that $\lambda_0 = 1$ and $\lambda_n = \lambda_{-n}$ decays exponentially at infinity.

Thus, in the case of the annulus, the spectrum of S determines the domain, up to homotheties and translations, in the family of all planar annuli.

The above computations can be reobtained in a more invariant way, starting with the generalized quadrature formula satisfied by the annulus:

$$\int_{A_t} f dA = \sinh t \int_{-\pi}^{\pi} f(e^{i\theta}) d\theta, \qquad f \in AL^2(A_t).$$

For details see [19] or [24]. The Bergman kernel of A_t can easily be obtained from orthonormalizing the system $z^n, n \in \mathbf{Z}$:

$$K_{A_t}(z, w) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{n+1}{\sinh(n+1)t} (z\overline{w})^n,$$

see for instance [15], Chapter VII, Section 5.

Thus, according to Theorem 5.5, an eigenfunction f of the operator S satisfies the equation:

$$(\sinh t)^2 \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} K(e^{iu}, e^{i\theta}) K(e^{iv}, e^{i\theta}) d\theta f(e^{iv}) dv = \lambda f(e^{iu}).$$

In particular one finds that the latter eigenvalue problem has the convolution form:

$$\int_{-\pi}^{\pi} G(u-v) f(e^{iv}) dv = \lambda f(e^{iu}).$$

More specifically, the evaluation of the integral in θ above yields:

$$G(u-v) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(n+1)\sinh t}{\sinh(n+1)t} \frac{(n-1)\sinh t}{\sinh(n-1)t} e^{in(u-v)}.$$

In conclusion, by Fourier transform, the eigenvalues of S are exactly the Fourier coefficients of the function G, including multiplicities. Thus formulas (29) are reproved.

6.3. **The ellipse.** The ellipse fulfills the conditions in Theorem 5.5. We show below how to identify the eigenfunctions of the Friedrichs operator by a more conceptual method than the original computations of Friedrichs [7].

Let $\mathbf{C}_n[z]$ be the space of polynomials of degree less than or equal to n.

Proposition 6.1. *The Friedrichs operator of an ellipse satisfies*

$$F\mathbf{C}_n[z] \subset \mathbf{C}_n[z], \qquad n \geq 0.$$

Proof. Let E be an ellipse and let p be a polynomial of degree n. Let $g = Fp$, so that $\overline{p} = g + G$, where $G \in L^2(E) \ominus AL^2(E)$. According to Havin's Lemma, [10]:

$$p = \overline{g} + \partial v / \partial \overline{z},$$

where the function v belongs to the Sobolev space $W_0^{1,2}(E)$. Let h be a holomorphic primitive of g, so that, integrating the last identity we obtain:

$$\overline{z}p = \overline{h} + v + H,$$

where $H \in AL^2(E) \cap C(\overline{E})$.

Therefore, $\overline{z}p(z) = \overline{h(z)} + H(z)$ for $z \in \partial E$. Since the solution of the Dirichlet problem on E with real polynomial data on ∂E is a real polynomial of the same total degree, or less (cf. [12]), it follows that $\partial^{n+2}(\overline{h} + H) = 0 = \overline{\partial}^{n+2}(\overline{h} + H)$, in E. Thus h is a polynomial of degree at most equal to $n+1$, and consequently g is a polynomial of degree at most n. □

Let E be an ellipse with foci at ± 1. Then E is a generalized quadrature domain with measure $d\mu = c(1-x^2)^{1/2} dx$, supported by the interval $[-1, 1]$, see [5]. According to Proposition 4.3, in this case $ker(F) = 0$. Thus $F\mathbf{C}_n[z] = \mathbf{C}_n[z]$ for all $n \geq 0$. The latter increasing chain of invariant subspaces diagonalizes F, or equivalently S, and this yields the following result.

Corollary 6.2. *The complete system of eigenfunctions of the Friedrichs operator of an ellipse with foci at ± 1 is provided by the Chebyshev polynomials of the second kind.*

In view of Theorem 5.5, the same eigenfunctions are shared by the modulus of the restriction operator R. Let us mention without giving here the details, that in this case the Bergman shift is represented by a non-selfadjoint Jacobi type matrix with only two non-zero diagonals (adjacent to the main diagonal, which is zero).

As for the computation of the corresponding eigenvalues of F we outline Friedrichs' computation, [7]. Let $E(t)$ be the ellipse centered at $z = 0$, with semiaxes $\cosh(t), \sinh(t)$, respectively. Let $U_n(\cos z) = \frac{\sin(n+1)z}{\sin z}$, $n \geq 0$, be the Chebyshev polynomials of the second kind, which we already know that are orthogonal on $E(t)$ and they are the eigenvectors of the corresponding Friedrichs operator, or better its square S. Thus the eigenvalues of $\sqrt{S}$ are:

$$\lambda_n(E(t)) = \frac{\int_{E(t)} U_n^2 dA}{\int_{E(t)} |U_n|^2 dA}, \qquad n \geq 0. \tag{30}$$

The numerator can be computed from the quadrature formula for the ellipse, cf. [24] formula (3.10):

$$\int_{E(t)} U_n^2 dA = \pi \cosh(t) \sinh(t) = (\pi/2) \sinh(2t).$$

Since the map $w = \cos(z)$ transforms conformally the rectangle $R(t) = \{z = x + iy;\ 0 < x < 2\pi,\ 0 < y < t\}$ onto $E(t)$ minus a segment of its major axis, the denominator in the above formula can be computed by a change of variables:

$$\int_{E(t)} |U_n(w)|^2 dA(w) = \int_{R(t)} |U_n(\cos z)|^2 |\sin z|^2 dA(z) =$$

$$[\pi/(2n+2)] \sinh(2(n+1)t).$$

Thus, returning to equation (24) we find:

$$\lambda_n(E(t)) = \frac{(n+1)\sinh 2t}{\sinh 2(n+1)t}, \qquad n \geq 0.$$

Note that the first eigenvalue is $\lambda_0(E(t)) = 1$ and that $\lambda_n(E(t))$ decays exponentially when n tends to infinity. The limiting case $t \to 0$ produces after rescaling a strip, which will be the subject of the next example. In that case the operator S has only continuous spectrum, equal to the interval $[0, 1]$, as this limiting process suggests.

6.4. **A strip.** Let c be a positive constant. The strip $\Omega = \{z = x + iy;\ |y| < c\}$ fulfills the conditions in Theorem 5.5. Indeed, it is known that:

$$\int_\Omega u dA = 2c \int_{-\infty}^{\infty} u(x, 0) dx, \qquad u \in HL^1(\Omega). \tag{31}$$

We outline the proof, which is needed again for the next example (the wedge). More details can be found in [20] (and the reference [Shapiro 1987] there). By approximation, it is enough to prove this quadrature identity for u fast decreasing at infinity and we can also assume that $u = \Re f$ with f analytic in Ω. By Cauchy's theorem, the integral $\int_{-\infty}^{\infty} f(x+iy)dx$ is independent of y, $|y| < c$. So,

$$\int_\Omega f dA = \int_{-c}^{c} \int_{-\infty}^{\infty} f(x+iy) dx dy = 2c \int_{-\infty}^{\infty} f(x) dx.$$

In order to use Theorem 5.5, we need the Bergman kernel of Ω. Without loss of generality we can work next with the strip $|y| < \pi/2$. Let $g : \Omega \longrightarrow \mathbf{D}$ be the conformal map which satisfies $g(0) = 0, g'(0) > 0$:

$$g(z) = \frac{e^z - 1}{e^z + 1}, \ z \in \Omega.$$

The kernel is therefore:

$$K(z,w) = \frac{1}{\pi} \frac{g'(z)\overline{g'(w)}}{(1 - g(z)\overline{g(w)})^2}.$$

Note in addition that the Bergman kernel enjoys the invariance property with respect to translations in the real direction: $K(z,w) = K(z+a, w+a)$ $a \in \mathbf{R}$. Therefore, for real entries x, ξ we obtain $K(x,\xi) = k(x-\xi)$, where k is a real function of a real variable. To find it, we take $\xi = 0$ and use the above formula:

$$k(x) = \frac{1}{\pi} g'(x) g'(0) = \frac{1}{\pi} \frac{e^x}{(e^x+1)^2} = \frac{1}{4\pi} \frac{1}{\cosh^2(x/2)}, \quad x \in \mathbf{R}.$$

According to Corollary 5.6 the operator $\sqrt{S}$ is unitarily equivalent to the integral operator:

$$(RR^*f)(x) = \frac{1}{4} \int_{-\infty}^{\infty} \frac{1}{\cosh^2((x-t)/2)} f(t)dt, \qquad f \in L^2(\mathbf{R}). \tag{32}$$

By taking the Fourier transform and using the known integral:

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{\cosh^2(x/2)} dx = \frac{4\pi\xi}{\sinh \pi\xi},$$

the operator (32) is unitarily equivalent to the multiplication operator:

$$\hat{f}(\xi) \mapsto \frac{\pi\xi}{\sinh \pi\xi} \hat{f}(\xi),$$

still in $L^2(\mathbf{R})$, by Plancherel theorem.

The multiplier $m(\xi) = \frac{\pi\xi}{\sinh \pi\xi}$ obviously satisfies the inequalities $0 < m(\xi) \leq 1$, $\xi \in \mathbf{R}$. In conclusion, using standard spectral theory for self-adjoint operators, cf. [18] Chapter VII, we have proved the following result.

Proposition 6.3. *The modulus $\sqrt{S}$ of the Friedrichs operator of a strip is unitarily equivalent to the operator $M_t \oplus M_t$, where M_t is the multiplication with the variable, acting on $L^2([0,1], dt)$.*

6.5. **A wedge.** Let $W_\alpha = \{z \in \mathbf{C}; |\arg z| < \alpha/2\}$ be a wedge of angle $0 < \alpha < 2\pi$. Denote $\beta = \pi/\alpha$. By using the conformal map:

$$\phi : W_\alpha \longrightarrow \mathbf{D}, \qquad \phi(z) = \frac{z^\beta - 1}{z^\beta + 1}, \quad z \in W_\alpha,$$

one finds the Bergman kernel of W_α:

$$K(z,w) = \frac{1}{\pi} \frac{\phi'(z)\overline{\phi'(w)}}{(1 - \phi(z)\phi(w))^2} = \frac{\beta}{\alpha} \frac{z^{\beta-1}\overline{w}^{\beta-1}}{(z^\beta + \overline{w}^\beta)^2}.$$

Equivalently, by using the automorphism group $z \to tz, t > 0$ of the wedge, one can start with the invariance formula $K(z,t) = \frac{1}{t^2} K(\frac{z}{t}, 1)$, $z \in W_\alpha, t > 0$ and reach by analytic continuation the same formula for K.

For a function $f \in AL^2(W_\alpha)$ rapidly decreasing at infinity, the transformation of the area integral in polar coordinates yields:

$$\int_{W_\alpha} f dA = \int_{-\alpha/2}^{\alpha/2} d\theta \int_0^\infty f(re^{i\theta}) r dr = \int_{-\alpha/2}^{\alpha/2} e^{-2i\theta} d\theta \int_{l_\theta} f(z) z dz,$$

where l_θ is the ray of angle θ. Since the function f decreases fast enough at infinity, Cauchy's theorem shows that the line integral along l_θ does not depend on θ. In this way one obtains the generalized quadrature formula satisfied by the wedge (see for more details [20]):

$$\int_{W_\alpha} f dA = \sin\alpha \int_0^\infty f(x) x dx, \qquad f \in AL^2(W_\alpha). \tag{33}$$

We mention that it is known that the above relation remains valid for $f \in AL^1(W_\alpha)$, see [24] and the references cited there.

According to Corollary 5.6, the spectrum of $\sqrt{S}$ is identical to the spectrum of the operator:

$$(RR^* f)(x) = \frac{\beta \sin\alpha}{\alpha} \int_0^\infty \frac{(xt)^{\beta-1}}{(x^\beta + t^\beta)^2} t f(t) dt. \tag{34}$$

The change of variables $x = e^u$, $t = e^v$ and the tranformation $F(u) = f(e^u)e^u$ maps $L^2([0,\infty), xdx)$ unitarily onto $L^2(\mathbf{R}, dx)$. The integral operator in (34) becomes:

$$\int_{-\infty}^{\infty} \frac{e^{(\beta-1)(u+v)}}{(e^{\beta u} + e^{\beta v})^2} f(e^u) e^{2u} du = e^{-v} \int_{-\infty}^{\infty} \frac{e^{\beta(u+v)}}{(e^{\beta u} + e^{\beta v})^2} F(u) du.$$

Thus the integral operator (34) is unitarily equivalent to:

$$F(u) \mapsto \frac{\beta \sin\alpha}{4\alpha} \int_{-\infty}^{\infty} \frac{F(u)}{\cosh^2 \beta \frac{u-v}{2}} du.$$

By passing to Fourier transforms, as in the previous example, we infer that $\sqrt{S}$ is unitarily equivalent to the multiplication operator:

$$\hat{F}(\xi) \mapsto \hat{F}(\xi) \frac{\sin\alpha}{\alpha} \frac{\pi\xi/\beta}{\sinh(\pi\xi/\beta)}.$$

Therefore, the following proposition holds.

Proposition 6.4. *The modulus $\sqrt{S}$ of the Friedrichs operator of the wedge W_α is unitarily equivalent to the operator*

$$\frac{\sin\alpha}{\alpha}(M_t \oplus M_t) : L^2([0,1], dt) \oplus L^2([0,1], dt) \longrightarrow L^2([0,1], dt) \oplus L^2([0,1], dt).$$

In conclusion, we stress the unitary equivalence:

$$(\frac{\sin\alpha}{\alpha})^2 S_\Omega \equiv S_{W_\alpha},$$

between the squares of the Friedrichs operators of a strip Ω and a wedge W_α (of angle α).

Final comments. Below we list a few subjects which will be expanded in the next article.

Various generalizations of the Friedrichs operator come now naturally into discussion. For instance the Friedrichs operator for the Hardy space of a domain with nice boundary appears as a "defect" operator in the corresponding Herglotz-Riesz formula; or a natural notion of Friedrichs operators with symbols is very much related to Hankel operators on the Bergman space.

As mentioned in the introduction, the connection between the Friedrichs operator and the boundary value problems for the bi-Laplacian or $\overline{\partial}^2$ is only touched in the present paper.

Most likely, the list of computable spectra of S_Ω's can be enlarged for domains with simple Bergman kernels and "mother bodies", (the latter in the terminology of [8]).

Part of the results and proofs above make sense in several complex variables. We leave the interested reader the easy task of finding these precise statements in $\mathbf{C}^n$. The specific, new features of the Friedrichs operator in several complex variables will be investigated elsewhere.

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