THE FRIEDRICHS OPERATOR OF A PLANAR DOMAIN. II

MIHAI PUTINAR AND HAROLD S. SHAPIRO

Paper dedicated to the memory of Béla Szőkefalvi-Nagy

ABSTRACT. We consider relations between the Friedrichs operator and constructive aspects of the Dirichlet problems for the Laplace and $\overline{\partial}^2$-operator. Then we investigate the Fourier expansions in the eigenfunctions of the Friedrichs operator. A link between a generalized Friedrichs operator and minimal nodes quadratures for complex polynomials of a fixed degree is explained. We initiate a discussion of the boundary Friedrichs operator, on the Hardy space of a domain. The transformation law of the Friedrichs operator under conformal mappings leads to a modified version of it, based on a symbol function; this object will turn out to be closely related to Hankel operators. We obtain some results concerning which symbols correspond to compact operators.

1. INTRODUCTION

Let Ω be a planar domain and let $AL^2(\Omega), CAL^2(\Omega)$ be the Bergman space and its complex conjugated space, in $L^2(\Omega, dA)$, where dA stands for the Lebesgue measure in $\mathbf{C}$. Motivated by some planar elasticity problems, Friedrichs [7], has studied the "gap operator" (in the sense of Kato [12] Chapter IV) between these subspaces, and a natural anti-linear square root of it F, called below the *Friedrichs operator* of the domain Ω. To be more specific, the operator F acts as follows:

$$\langle f, Fg \rangle = \langle g, Ff \rangle = \int_\Omega fg dA, \qquad f, g \in AL^2(\Omega).$$

A first paper [25] was devoted to some basic spectral theory questions for the operator F, and their relevance to the geometry and analysis on Ω. The present note is a direct continuation of [25], touching other aspects of this rich field.

The contents are as follows. In Section 2, for a Jordan domain with rectifiable boundary one starts with a function f in the Hardy space $H^2(\Omega)$ and obtains the decomposition $\overline{z}f(z) = \overline{\int_{z_0}^z Ff(u)du} + h(z)$, with $h \in H^2(\Omega)$ and $z_0 \in \Omega$. This simple fact has remarkable consequences: for instance if Fz is a linear polynomial, then Ω must be an ellipse. Or a solution of the Dirichlet problem with boundary data of the form $\overline{z}^n f(z)$ can recurrently be described in terms of the action of the Friedrichs operator on such special

Partially supported by the National Science Foundation Grant DMS- 9800666.

functions, and quadratures. Similarly one obtains necessary conditions for the existence of bi-analytic functions with prescribed boundary data, see also [2] and [5] for related topics.

Section 3 deals with some simple aspects of the Fourier series expansions in the eigenfunctions of the operator $\sqrt{S}$, where $S = F^2$. Whenever the *Friedrichs inequality* holds, see Theorem 4.1 for the precise statement, one can prove that any square summable harmonic function in the simply connected domain Ω is the sum of a square summable analytic function and the conjugate of such a function. This opens a natural way of computing the reproducing kernel for the space of square summable harmonic functions. This is sketched in Section 4.

Section 5 is based on the remark that, given a degree n, to find minimal quadrature formulas on a bounded domain Ω, for all complex polynomials of degree less or equal than n, involves only the bilinear form $E(p,q) = \langle p, Fq \rangle = \int_{\Omega} pqdA$. A canonical space which parametrizes all these minimal quadratures is described in terms of n and the action of a generalized Friedrichs operator on polynomials (of degree less than n).

An extension to Friedrichs operators with L^{∞}-symbol is considered in Section 6. This is motivated, for instance, by the pull back of the Friedrichs operator via a conformal map. The compactness of such an operator is discussed with the methods of Chapter 8 of [30]. In particular we obtain another proof of Friedrichs' result on the essential spectrum of F on a domain with corners.

The analogous Friedrichs operators with symbols, acting on the Hardy space of a domain with rectifiable boundary are defined in Section 7. On the unit disk, treated in Section 8, these operators are rank-one perturbations of some conjugated Hankel operators. For completeness we include the well known Nehari's compactness criterion, adapted to the setting of Friedrichs operators.

We are indebted to the two anonymous referees, whose accurate observations and recommendations led to an improved version of this paper.

Notation and terminology (consistent with those in [25]).

Throughout this paper Ω will be a planar domain; the regularity assumptions on its boundary $\partial\Omega$ will be made clear in each section separately. The area measure in $\mathbf{C}$ will be denoted by dA and the associated Lebesgue spaces by $L^p(\Omega) = L^p(\Omega, dA)$. We put $W^{s,p}(\Omega)$, $\overset{\circ}{W}{}^{s,p}(\Omega)$ for the Sobolev spaces of functions, or distributions, in Ω having the derivative up to order s in $L^p(\Omega, dA)$, respectively the closure of the Schwartz space $D(\Omega)$ in $W^{s,p}(\Omega)$. The Bergman space associated to Ω will be denoted by $AL^2(\Omega)$ and the Bergman projection by P, that is the orthogonal projection of $L^2(\Omega) = L^2(\Omega, dA)$ onto $AL^2(\Omega)$. The Hardy spaces of a domain Ω with sufficiently regular boundary will be denoted by $H^p(\Omega)$, $1 \leq p \leq \infty$. The arc length element on $\partial\Omega$ will be ds.

Let $C : L^2(\Omega) \longrightarrow L^2(\Omega)$ be the complex conjugation operator: $Cf = \overline{f}, f \in L^2(\Omega)$. The *Friedrichs operator* of the domain Ω is then the anti-linear operator $F = PC : AL^2(\Omega) \longrightarrow AL^2(\Omega)$; we put as before $S = F^2$, so that S is a complex linear, non-negative and contractive operator. Throughout this paper we refer to $\sqrt{S}$ as the *modulus* of the Friedrichs operator F.

2. The Friedrichs operator and certain boundary value problems

We show that the Friedrichs operator of a planar domain is intimately connected to the Dirichlet problems for the Laplacian and for the square of the Cauchy-Riemann operator. The main observation we develop below is that, knowing the action of the Friedrichs operator on a space of functions f, such as the complex polynomials, it is a matter of finitely many quadratures to solve the Dirichlet problem for Δ with boundary data $\overline{z}^n f(z)$, for arbitrary $n \geq 1$. Then a solution to the $\overline{\partial}^2$ Dirichlet problem is also at hand.

Throughout this section we assume that Ω is a Jordan domain with rectifiable boundary $\Gamma = \partial\Omega$, which moreover is sufficiently regular that the space $W^{1,2}(\Omega)$ has traces in $L^2(\Gamma, ds)$. For some purposes even more regularity will be needed. In this initial study we shall not strive for maximal generality. Therefore we shall sometimes invoke a rather vague hypothesis: "sufficiently smooth", leaving for future study the sharpest possible results. Certainly, C^2 regularity is more than enough to justify all our assertions.

The next proposition is a simple application of the
Friedrichs-Havin Lemma, see [7], [11].

Proposition 2.1. *Let $f \in H^2(\Omega)$, where $\partial\Omega$ is sufficiently smooth, and let G be a primitive function of $g = Ff$. Then*

$$\overline{\zeta} f(\zeta) = \overline{G(\zeta)} + h(\zeta), \qquad \zeta \in \Gamma \quad (ds - a.e.), \tag{1}$$

where $h \in H^2(\Omega)$.

Proof. By definition, the function $g - \overline{f}$ is orthogonal to $AL^2(\Omega)$. So, by Friedrichs-Havin lemma, there exists a distribution $v \in \overset{\circ}{W}{}^{1,2}(\Omega)$ such that $\overline{\partial} v = \overline{g} - f$, in Ω, that is:

$$\overline{\partial}[\overline{z} f(z) - \overline{G(z)} + v(z)] = 0, \qquad z \in \Omega.$$

Thus there exists a holomorphic function $h(z)$, $z \in \Omega$, with $h(z) = \overline{z} f(z) - \overline{G(z)} + v(z), z \in \Omega$. Our regularity assumptions imply that G, h are in $H^2(\Omega)$, hence, by taking traces on Γ and using the fact that v vanishes there a.e., we obtain relation (1). □

Note that the decomposition (1) can be reversed and used in computing the value Ff of the Friedrichs operator from the solution u of the Dirichlet problem with boundary data $\overline{\zeta} f(\zeta)$. Indeed, the harmonic function u can always be decomposed, uniquely up to an additive constant, as $\overline{G} + h$, with G, h in the Hardy space $H^2(\Omega)$, and then $Ff = G'$.

The decomposition (1) has several interesting consequences. First of all we note the following regularity result.

Theorem 2.2. *Let Ω be a Jordan domain with sufficiently smooth boundary $\Gamma = \partial\Omega$, so that the space $W^{1,2}(\Omega)$ has traces in $L^2(\Gamma, ds)$.*

If Fz is the derivative of a rational function, then Γ is a subset of an algebraic curve.

Proof. Write $\phi = Fz$ and let Φ be a primitive of ϕ, assumed to be rational. Thus relation (1) yields

$$\overline{\zeta}\zeta = \overline{\Phi(\zeta)} + h(\zeta), \qquad \zeta \in \Gamma, \ a.e., \tag{2}$$

hence the function $h - \Phi \in H^2(\Omega)$ has boundary values $\overline{\zeta}\zeta - \overline{\Phi(\zeta)} - \Phi(\zeta)$, $\zeta \in \Gamma$, which are real. Therefore $h - \Phi$ is constant, and we can write:

$$|\zeta|^2 - 2\Re\Phi(\zeta) = c, \qquad \zeta \in \Gamma,$$

where c is a real constant. It remains to remark that the function $R(z, \overline{z}) = |z|^2 - 2\Re\Phi(z) - c$ is not constant since its Laplacian equals 4. □

Inspection of the above proof shows that the simple assumption that Fz extends analytically across the boundary of Ω implies that the latter is real analytic, locally or globally.

As an application of the reasoning used to prove Theorem 2.2 we state the following characterization of ellipses, including among them the disks.

Corollary 2.3. *In the conditions of Theorem 2.2, assume that Fz is a linear polynomial. Then Ω is an ellipse.*

Proof. In this situation $\Phi(\zeta)$ is a quadratic polynomial, and by invoking the proof of the theorem, the boundary Γ of Ω is contained in the set of zeroes of a real polynomial in $z, \overline{z}$, of total degree 2. Hence Ω must be an ellipse. □

By combining this corollary with the analysis of the eigenvalues of the Friedrichs operator of an ellipse, cf. [25] Proposition 6.1, we obtain the following result. Below we denote by P_n the space of complex polynomials of degree less or equal than n.

Corollary 2.4. *Let Ω be Jordan domain as in Theorem 2.2. The following conditions are equivalent:*

a). The space P_1 is invariant under F;

b). For all $n \geq 1$, the space P_n is invariant under F;

c). Ω is an ellipse.

Next we turn to the Dirichlet problem for the Laplace operator. Let f, G, h be as in Proposition 2.1, and let $z \in \Omega$. Then, since $h \in H^2(\Omega)$:

$$h(z) = \frac{1}{2\pi i}\int_\Gamma \frac{h(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\int_\Gamma \frac{\overline{\zeta f(\zeta)} - \overline{G(\zeta)}}{\zeta - z} d\zeta. \tag{3}$$

Consequently we can state the following observation.

Corollary 2.5. *The solution to the Dirichlet problem:*

$$\Delta u = 0 \quad in \quad \Omega; \quad u(\zeta) = \overline{\zeta} f(\zeta), \ \zeta \in \Gamma,$$

is $u(z) = \overline{G(z)} + h(z)$, *where the function* h *is defined by (3).*

Observe that, once $g = Ff$ is known, both G and h are determined by integrations. This opens the way to recursively solving the Dirichlet problem with data of the form $\overline{z}^m f(z)$, with $m \geq 0$ and $f \in H^2(\Omega)$, in terms of the Friedrichs operator and finitely many integration operations. Indeed, starting from (1) we obtain:

$$\overline{\zeta}^2 f(\zeta) = \overline{\zeta G(\zeta)} + \overline{\zeta} h(\zeta), \qquad \zeta \in \Gamma.$$

Since the function $zG(z)$ is analytic, solving the Dirichlet problem with boundary data $\overline{\zeta}^2 f(\zeta)$ is reduced to that for $\overline{\zeta} h(\zeta)$, and hence to a second application of Corollary 2.5. In particular, choosing $f(z) = z^k, k \geq 1$, we see that the solution of the Dirichlet problem with polynomial data on the boundary is reduced to the knowledge of the Friedrichs operator and a finite number of integrations. This is of special interest in quadrature domains, when F has finite rank and in principle can be calculated once its values are known on a finite set of functions, see [25].

To make the above remark more precise, let us define the operators A, B, acting on the Hardy space $H^2(\Omega)$ as:

$$(Af)(z) = \int_{z_0}^{z} (Ff)(w) dw, \qquad (Bf)(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\overline{\zeta} f(\zeta) - \overline{Af(\zeta)}}{\zeta - z} d\zeta,$$

where z_0 is a fixed point in Ω and the integration in the expression of A is taken on a path fully contained in Ω. The operators

$$CA : H^2(\Omega) \longrightarrow CH^2(\Omega), \qquad B : H^2(\Omega) \longrightarrow H^2(\Omega)$$

are complex linear and bounded on the Hardy space and they satisfy, in the sense of boundary values, the identity:

$$\overline{\zeta} f(\zeta) = CAf(\zeta) + Bf(\zeta), \qquad \zeta \in \Gamma. \tag{4}$$

By iterating this relation we obtain, for an arbitrary polynomial $P(z)$, the next formula, in which the standard notation $P^*(z) = \overline{P(\overline{z})}$ appears:

$$\overline{P(\zeta)} f(\zeta) = [P^*(B) f](\zeta) + C[A \frac{P(z) - P(B)}{z - B} f](\zeta), \qquad \zeta \in \Gamma.$$

The symbolic notation $\frac{P(z)-P(B)}{z-B}$ means the value of the polynomial function $\frac{P(z)-P(w)}{z-w}$ at $w = B$. The first term in the decomposition above extends analytically inside Ω, while the second term, understood as a polynomial in z, A and B, with the multiplication operator by z acting to the *left* of A and B, admits an anti-analytic extension in Ω. Therefore, we are led to the following "explicit" formula for solving the Dirichlet problem.

Corollary 2.6. *Let $P(z)$ be a polynomial and let $f \in H^2(\Omega)$. The solution to the Dirichlet problem*

$$\Delta u(z) = 0, \quad z \in \Omega; \quad u(\zeta) = \overline{P(\zeta)} f(\zeta), \quad \zeta \in \Gamma,$$

is

$$u(z) = [P^*(B)f](z) + C[A\frac{P(z) - P(B)}{z - B} f](z). \tag{5}$$

Finally, let us remark that formula (5) can be extended to more general functions instead of polynomials, as soon as the functional calculus $P(B)$ and the evaluations $P(z), z \in \Omega \cup \Gamma$, are well defined (as for instance for entire functions).

Another application of decomposition (1) is to the $\overline{\partial}^2$ operator. Suppose we wish to solve the boundary value problem:

$$\overline{\partial}^2 u = 0 \ \ in \ \ \Omega; \quad u(\zeta) = p(\zeta), \quad \zeta \in \Gamma, \tag{6}$$

for some function p on Γ.

The most general solution to $\overline{\partial}^2 u = 0$ is of the form $u(z) = f_1(z) + \overline{z} f_2(z)$, with f_1, f_2 analytic functions in Ω. If we suppose the Dirichlet problem for the Laplacian with boundary data p solved, then we can write $p = a + \overline{b}$, with a, b analytic in Ω. Then clearly (6) reduces to $p = \overline{b}$, and the decomposition (1) just solves this problem. To be more specific we have proved, modulo the regularity details which are left to the reader, the following proposition.

Proposition 2.7. *Assume that $p \in L^2(\Gamma, ds)$. Then problem (6) is solvable if and only if $p = \overline{G}$, where G is the primitive of an element $g \in Ran(F)$, $g = Ff$, and in this case the solution is $u(z) = [(\overline{z} - B)f](z)$.*

Thus, the $\overline{\partial}^2$ Dirichlet problem can be reduced to the Laplace Dirichlet problem and knowledge of the Friedrichs operator. Again, in quadrature domains this reduces to finitely many quadratures and algebraic computations, modulo a fixed finite "table" of values of F.

3. Fourier expansions in the Friedrichs eigenfunctions

The aim of this section is to continue the study, started in [25], of the elements in the kernel and the range of the Friedrichs operator.

First we derive from Proposition 2.1 a characterization of quadrature domains in terms of the location of zeroes of functions in the kernel of F.

Proposition 3.1. *Let Ω be a bounded domain with rectifiable boundary Γ. Then Ω is a quadrature domain if and only if there exists a function $f \in H^2(\Omega)$ satisfying $Ff = 0$ and such that there exists $\epsilon > 0$ with $|f(z)| > \epsilon$ for $dist(z, \Gamma) < \epsilon$.*

Proof. If Ω is a quadrature domain of order d, then the Friedrichs operator F has rank d, [25]. We also know that any element $f \in \ker F$ vanishes, counting multiplicities, at the quadrature nodes. Since a non-trivial linear combination of the monomials $1, z, z^2, \dots, z^d$ lies in $\ker F$, there exists a monic polynomial p of degree d, which satisfies: $Fp = 0$ and p is zero on d points belonging to Ω. Consequently $p(z)$ is bounded away from zero on $\partial\Omega$.

Conversely, assume that the function f as in the statement exists. Then, according to decomposition (1) we obtain:

$$\overline{\zeta} f(\zeta) = h(\zeta), \quad \zeta \in \Gamma,$$

with $h \in H^2(\Omega)$. Consequently, the function $\overline{\zeta}$ admits the meromorphic extension $h(z)/f(z)$ throughout Ω, and this is equivalent to Ω being a quadrature domain, see [30]. □

A consequence of the previous proof is that, functions $f \in \ker(F)$ which are bounded away from zero in a neighbourhood of the boundary Γ must have zeroes in Ω.

Also, the *apriori* weaker hypothesis that to each point $\zeta \in \partial\Omega$ there exists $f \in \ker F$ and $\epsilon > 0$ so that $|f(z)| > \epsilon$ for all $z \in \Omega \cap D(\zeta, \epsilon)$ suffices to show that Ω is a quadrature domain.

Next we consider a bounded domain Ω with compact and injective Friedrichs operator F, so that the same is true for the square $S = F^2$. (For examples and necessary conditions for these assumptions, see [25].) Let $1 = \lambda_0 \geq \lambda_1 \geq \dots$ be the eigenvalues of $\sqrt{S}$, with associated eigenfunctions $\phi_n \in AL^2(\Omega)$, $n \geq 1$, which form an orthonormal basis of the Bergman space.

The following general criterion is well known for any compact operator and we omit its proof.

Lemma 3.2. *The necessary and sufficient condition for a function $g \in AL^2(\Omega)$ to belong to the range of F is:*

$$\sum_{n=0}^{\infty} {\lambda_n}^{-2} |\langle g, \phi_n \rangle|^2 < \infty. \tag{7}$$

In the specific situation of the Friedrichs operator this leads to the next proposition.

Proposition 3.3. *With assumptions as above,*

$$\sum_{n=0}^{\infty} \frac{|\phi_n(z)|^2}{{\lambda_n}^2} = \infty, \tag{8}$$

for each $z \in \Omega$.

Proof. In view of Lemma 3.2 , the finiteness of the left hand side of (8) would imply that the Bergman evaluation function k_z belongs to the range of F, and we know from [25] Corollary 4.2 that this cannot happen. □

The same idea leads to more general results, as follows.

Proposition 3.4. *Assume F compact and let u be a distribution $u \in L^2(\Omega)$ satisfying $fu = 0$ for a nontrivial element $f \in H^\infty(\Omega)$. Then:*

$$\sum_{n=0}^{\infty} \frac{|u(\phi_n)|^2}{{\lambda_n}^2} = \infty. \tag{9}$$

Proof. Recall from [25] that the Toeplitz operator T_f corresponding to f satisfies $FT_f = T_f^* F$. Let a be the representing vector for u:

$$u(g) = \langle g, a \rangle, \quad g \in AL^2(\Omega).$$

Then, by our assumption, $T_f^* a = 0$.

Assume by contradiction that $a = Fh$ for some $h \in AL^2(\Omega)$. Then $FT_f h = T_f^* Fh = T_f^* a = 0$, which contradicts the injectivity of F, because $T_f h = fh \neq 0$. □

4. The harmonic kernel

This section is mainly devoted to the computation of the orthogonal projection of the Lebesgue space $L^2(\Omega, dA)$ onto the closed subspace $HL^2(\Omega)$ of square-summable harmonic functions in Ω. We prove that, if Ω is simply connected, then whenever the Friedrichs inequality holds, the space $AL^2(\Omega) + CAL^2(\Omega)$ is closed and equals $HL^2(\Omega)$. In addition, a general formula due to Lenard, [16], gives a rather explicit formula for the orthogonal projection onto the sum of two non-orthogonal subspaces.

First we recall some basic facts and conventions concerning Friedrichs' inequality. For more details we refer to [29], [30] and [25].

Theorem 4.1. *(H. S. Shapiro [29]). Let Ω be a bounded domain satisfying an interior cone condition and let $z_0 \in \Omega$. Let $f \in AL^2(\Omega)$ be a function subject to one of the following normalizations:*

$$a). \quad \int_\Omega f dA = 0, \quad or \quad b). \quad f(z_0) = 0.$$

Then the Friedrichs inequality holds in either of the forms:

$$|\int_\Omega f^2 dA| \leq c \int_\Omega |f|^2 dA, \tag{10}$$

for some constant $c < 1$, or:

$$\int_\Omega |\tilde{u}|^2 dA \leq C \int_\Omega |u|^2 dA, \tag{11}$$

for some finite constant C, where $u = \Re f$ and $\tilde{u} = \Im f$.

The constants c, C above depend only on Ω if the normalization a) is used, and respectively on Ω and z_0 under the normalization b). Inequality (10) holds also if f^2 is replaced by an arbitrary function in $AL^1(\Omega)$ normalized to have mean zero, or to vanish at z_0. Moreover, (11) holds also for exponents $p > 1$, and then C depends also on p. Finally, in (11) it is sufficient to normalize $\tilde{u}$ by one of the conditions a) or b).

For the sake of completeness we state below some consequences of the Friedrichs inequality, part of them previously known in one form or another.

Corollary 4.2. *If Ω satisfies an interior cone condition, and is simply connected, then for every real valued $u \in HL^2(\Omega)$, its harmonic conjugate satisfies $\tilde{u} \in HL^2(\Omega)$.*

Proof. We only sketch the idea of the proof. One can show that Ω is the union of an increasing sequence Ω_j, $j \geq 1$, of simply connected domains which uniformly satisfy an interior cone condition. Let z_0 be a point of Ω_1. We can assume that $\tilde{u}(z_0) = 0$. Then, by the Friedrichs inequality (11) the norms $\|\tilde{u}\|_{2,\Omega_j}$ are uniformly bounded in j, hence $\tilde{u} \in L^2(\Omega)$. □

Corollary 4.3. *If Ω is simply connected, then every $u \in HL^2(\Omega)$ can be written, uniquely up to an additive constant, as $f + \overline{g}$, with $f, g \in AL^2(\Omega)$.*

Proof. It suffices to deal with a real valued u. By the preceding corollary the harmonic conjugate function $\tilde{u}$ is square integrable in Ω, hence $2u = f + \overline{f}$, where $f = u + i\tilde{u} \in AL^2(\Omega)$. □

Corollary 4.4. *In any domain Ω for which the Friedrichs inequality holds (in particular, for Ω satisfying an interior cone condition) we have, whenever $f, g \in AL^2(\Omega)$ and g satisfies one of the normalizations in Theorem 4.1:*

$$\|f\| + \|\overline{g}\| \leq C\|f + \overline{g}\|, \tag{12}$$

where the norm is in $L^2(\Omega)$ and C is a constant depending only on Ω (or Ω and z_0, as the normalization demands).

Proof. In virtue of Friedrichs' inequality and Schwarz' inequality we obtain:

$$\|f + \overline{g}\|^2 = \|f\|^2 + \|g\|^2 + 2\Re \int_\Omega fgdA \geq$$

$$\|f\|^2 + \|g\|^2 - 2c \int_\Omega |fg|dA \geq$$

$$(1-c)[\|f\|^2 + \|g\|^2] + c(\|f\| - \|g\|)^2.$$

Since the last term is non-negative, relation (12) follows with $C = \sqrt{\frac{1}{1-c}}$. □

Corollary 4.5. *With the same hypotheses on Ω, analytic polynomials are dense in $AL^2(\Omega)$ if and only if harmonic polynomials are dense in $HL^2(\Omega)$.*

Proof. If analytic polynomials are dense in the Bergman space, then any $u \in HL^2(\Omega)$ can be decomposed as before $u = f + \overline{g}$ and therefore it is approximable by harmonic polynomials.

Conversely, assume that the harmonic polynomials are dense in $HL^2(\Omega)$ and let $f \in AL^2(\Omega)$ be given. Let $\epsilon > 0$ be arbitrary. Then there exists a harmonic polynomial h with the property $\|f - h\| < \epsilon$. Now, $h = f_1 + \overline{g_1}$, where f_1, g_1 are analytic polynomials and, without loss of generality $g_1(z_0) = 0$ for some fixed point z_0. Thus, $\|(f - f_1) - \overline{g_1}\| \leq \epsilon$, and, using (12),

$$\|f - f_1\| \leq C\epsilon,$$

which completes the proof. □

The observation that P is the orthogonal projection onto $AL^2(\Omega)$, $Q = CF = CPC$ is the orthogonal projection onto $CAL^2(\Omega)$, and $PQ = S = F^2$, leads to the following simple but useful partial conclusion of the above results. For details about the geometric relations of two subspaces of a Hilbert space we refer to [12], Section IV.2.

Proposition 4.6. *Let Ω be a bounded planar domain. Then the following assertions are equivalent:*

1). The Friedrichs inequality holds;

2). Restricted to the subspace of $AL^2(\Omega)$ consisting of functions with mean value zero, the operator S has norm less than 1;

3). The space $AL^2(\Omega) + CAL^2(\Omega)$ is closed.

In addition, if Ω is simply connected, then the above assertions are equivalent to:

4). The vector sum $AL^2(\Omega) + CAL^2(\Omega)$ equals $HL^2(\Omega)$.

Our next aim is to compute the reproducing kernel $H(z, w)$ of the space $HL^2(\Omega)$, under the assumption that the domain Ω is simply connected and satisfies Friedrichs inequality.

In general, if P and Q are two orthogonal projections of a Hilbert space H, such that, for instance $\|PQ\| < 1$, then $Ran(P) \cap Ran(Q) = 0$, the subspace $V = Ran(P) + Ran(Q)$ is closed and the operators $I - PQ, I - QP$ are invertible. By a formula due to Lenard, [16], the orthogonal projection R onto V is :

$$R = (I - Q)(I - PQ)^{-1}P + (I - P)(I - QP)^{-1}Q.$$

We could simply use this formula, for the subspace H of functions $f \in L^2(\Omega)$ having mean 0 (so that $Ran(P) \cap Ran(Q) = 0$), and deduce the form of the harmonic reproducing kernel $H(z, w)$. Below we denote $AL^2_0(\Omega) = \{f \in AL^2(\Omega);\ \int_\Omega f dA = 0\}$.

As a matter of fact, assuming the spectral decomposition of the Friedrichs operator known, our situation is simpler.

Assume first that F is compact and injective, and let $1 = \lambda_0 > \lambda_1 \geq \lambda_2 \geq \ldots$ be the eigenvalues of $\sqrt{S}$ with corresponding eigenfunctions $\phi_n, n \geq 0$. The system of functions:

$$\psi_n = \frac{\overline{\phi_n} - \lambda_n \phi_n}{\sqrt{1 - \lambda_n^2}}, \quad n \geq 1, \tag{13}$$

is orthonormal and, by the very definition of F, $\langle \phi_m, \psi_n \rangle = 0$ for all $m \geq 1$ and $n \geq 0$. Thus $\phi_0, \phi_n, \psi_n, n \geq 1$ is an orthonormal basis of $HL^2(\Omega)$. As a first conclusion of these simple remarks we note the following lemma.

Lemma 4.7. *Assume that Ω is a simply connected domain which satisfies Friedrichs' inequality. Let $\phi_n, n \geq 0$, be the eigenfunctions of the modulus of the Friedrichs operator and let $\psi_n, n \geq 1$, be defined as in (13).*
Then the reproducing kernel of $HL^2(\Omega)$ is:

$$H(z,w) = \phi_0(z)\overline{\phi_0(w)} + \sum_{n=1}^{\infty}(\phi_n(z)\overline{\phi_n(w)} + \psi_n(z)\overline{\psi_n(w)}). \tag{14}$$

Second, we assume that Ω is a quadrature domain, so that, $1 = \lambda_0 > \lambda_1 \geq \lambda_2 \geq \ldots \lambda_d > \lambda_{d+1} = 0$. Then the finite dimensional space $Ran(F)$ is spanned by the vectors $\phi_n,\ n \leq d$, and the Bergman kernel $K(z,w)$ of Ω can be written as:

$$K(z,w) = N(z,w) + \sum_{n=0}^{d} \phi_n(z)\overline{\phi_n(w)}, \tag{15}$$

where $N(z,w)$ is the reproducing kernel of the space $Ker(F)$.

The harmonic space can be decomposed into an orthogonal sum as:

$$HL^2(\Omega) = [Ker(F) + CKer(F)] \oplus [Ran(F) + CRan(F)].$$

Moreover, the reproducing kernel of the first summand is precisely $N(z,w) + N(w,z)$, because, $Ff = 0$ implies

$$\int_\Omega f(z)(N(z,w) + N(w,z))dA = 0 + f(w),$$

and by symmetry,

$$\int_\Omega \overline{f(z)}(N(z,w) + N(w,z))dA = \overline{f(z)} + 0.$$

In conclusion we can state the following result.

Lemma 4.8. *If Ω is a quadrature domain, then the reproducing kernel of $HL^2(\Omega)$ is:*

$$H(z,w) = K(z,w) + K(w,z) - \phi_0(w)\overline{\phi_0(z)} + \sum_{n=1}^{d}(\psi_n(z)\overline{\psi_n(w)} - \phi_n(w)\overline{\phi_n(z)}). \tag{16}$$

Further on, recall ([25]) that ϕ_n is a linear combination of the evaluation vectors $K(z, a_i), 1 \leq i \leq d$, where $a_i, 1 \leq i \leq d$, are the quadrature nodes. Thus the real kernel H has the form:

$$H(z,w) = 2\Re K(z,w) + \Re \sum_{i,j=1}^{d} K(z,a_i)C_{ij}K(a_j,w) +$$

$$\Re \sum_{i,j=1}^{d} K(z,a_i) D_{ij} K(w,a_j),$$

with the unknown matrices C_{ij}, D_{ij}, to be determined by a linear system of reproducing formulas for the functions $f = K(z,a_i)$ and their complex conjugates. We do not expand here these details.

5. Interpolatory quadrature formulas

Since the Friedrichs operator encodes the geometrical relationship of two subspaces, it is a flexible concept, and can usefully be introduced in other contexts than that of Friedrichs. From this point on we explore such generalizations. We emphasize that in the following sections the term "Friedrichs operator" and symbol F thus get a special meaning. We hope this will cause no confusion.

A generalized Friedrichs operator is related below to Gaussian cubatures for complex polynomials. This is a direct extension of the relation between orthogonal polynomials and Gaussian cubatures on the line.

To be more specific, let μ be a positive measure on $\mathbf{C}$, rapidly decreasing at infinity, and with infinite support. These conditions assure that any complex analytic polynomial p is integrable with respect to μ, and $\int_{\mathbf{C}} |p| d\mu = 0$, if and only if $p = 0$. Let P denote the orthogonal projection of $L^2(\mu)$ onto the closure $P^2(\mu)$ of all polynomials. The Friedrichs operator F is then well defined:

$$Ff = PCf = P(\overline{f}), \quad f \in P^2(\mu).$$

Note that $\mathbf{C}[z] \subset L^2(\mu)$. For a fixed integer $d \geq 0$ we adopt the following notation:

$$P_d = \{p \in \mathbf{C}[z]; \;\; deg(p) \leq d\}, \quad FP_d = \{Fp; \;\; p \in P_d\}.$$

We will show that the angles between the elements of the increasing chains of subspaces $(P_d)_{d=0}^{\infty}$ and $(FP_d)_{d=0}^{\infty}$ are responsible for the existence of quadrature formulas of prescribed degree, for the measure μ.

Fix a degree $n \geq 1$. A *quadrature formula* of degree n, for the measure μ, has the form:

$$\int_{\mathbf{C}} p d\mu = \sum_{k=1}^{m} \sum_{j=1}^{n_k - 1} c_k^j p^{(j)}(a_k), \quad p \in P_n.$$

The number $N = \sum_{k=1}^{m} n_k$ is called the *order* of the quadrature formula.

A simple linear dependence argument shows that always, for fixed n, such formulas exist. We will be interested in minimal quadrature formulas, that is those with minimal order N. In this case we can assume that the formula is also *interpolatory*, that is the map:

$$P_n \longrightarrow \mathbf{C}^N, \quad p \mapsto (p^{(j)}(a_k))_{j,k},$$

is onto. Details about quadrature, or after some authors cubature, formulas can be found in [19]. Some aspects of the related theory of complex orthogonal polynomials is developed in [31].

We start with a series of simple remarks. For every d, $\dim P_d = d+1$, while $\dim FP_{d+1} \leq \dim FP_d + 1$. Indeed, denoting by M the multiplication operator on $\mathbf{C}[z]$ by the complex variable z and by $\mathbf{1}$ the functions equal to 1, we have:

$$FP_d = \{p(M^*)\mathbf{1};\ p \in P_d\}. \tag{17}$$

We have to be careful with the possible unboundedness of M on the whole space $\mathbf{C}[z]$; however, for arbitrary polynomials $p, q \in \mathbf{C}[z]$, the definition of M^* is unambigous:

$$\langle Mp, q\rangle = \langle p, M^*q\rangle = \langle p, P(\overline{z}q)\rangle.$$

Assume that for a given d, $\dim FP_{d+1} = \dim FP_d$. Then, according to formula (17), the subspace FP_d is invariant under the linear operator M^*. In virtue of the same formula we find that $FP_{d+k} = FP_d$ for all $k \geq 0$. Thus the second tower $(FP_d)_d$ is stationary. Let $T = M^*|_{FP_d}$, regarded as an operator from FP_d to itself. Let

$$\sigma(T) = \{\overline{a_1}, \overline{a_2}, \ldots, \overline{a_m}\}$$

be the spectrum of T. Let v_k be the corresponding generalized eigenvectors:

$$(T - \overline{a_k})^{n_k} v_k = 0.$$

In virtue of the fact that T admits $\mathbf{1}$ as a cyclic vector, for each a_k there exists exactly one such v_k. A basis of FP_d is then given by the vectors $(T - \overline{a_k})^j v_k,\ 1 \leq k \leq m, 0 \leq j \leq n_k - 1$. By decomposing $\mathbf{1}$ in this basis:

$$\mathbf{1} = \sum_{j,k} \overline{d_k^j} (T - \overline{a_k})^j v_k,$$

we obtain a universal quadrature formula:

$$\int_{\mathbf{C}} p d\mu = \langle p, \mathbf{1}\rangle = \sum_{j,k} d_k^j \langle p, (T - \overline{a_k})^j v_k\rangle.$$

After rearranging the terms above, we get:

$$\int_{\mathbf{C}} p d\mu = \sum_{j,k} c_k^j p^{(j)}(a_k), \quad p \in \mathbf{C}[z]. \tag{18}$$

Thus we can state the following proposition.

Proposition 5.1. *Let μ be a positive, rapidly decreasing measure on* $\mathbf{C}$. *The following assertions are equivalent:*

a). The measure μ admits a universal quadrature formula (18);

b). There exists a degree d_0 with the property that $FP_{d_0} = FP_{d_0+1}$;

c). The Friedrichs operator F annihilates a non-trivial polynomial.

Let dA be the planar Lebesgue measure, let Ω be a bounded quadrature domain and let χ_Ω be the characteristic function of Ω. Then the measure $\chi_\Omega dA$ satisfies the conditions in the above proposition, see [25]. More generally, assume that $\Omega_k, 1 \le k \le m$, are quadrature domains and that $\alpha_k > 0, 1 \le k \le m$. Then the measure $\mu = \sum_{k=1}^m \alpha_k \chi_{\Omega_k}$ still satisfies the conditions in Proposition 5.1.

From now on we assume that $dimFP_d = d+1$ for all $d \ge 0$. Assume that the quadrature formula of degree n is given. We denote by I_d the intersection of the polynomial ideal I vanishing at the quadrature nodes with P_d:

$$I_d = \{p \in P_d;\ \ p^{(j)}(a_k) = 0,\ 1 \le k \le m, 0 \le j \le n_k - 1\}.$$

Note that for all $d \ge 0$, $I_{d+1} \cap P_d = I_d$, hence the maps induced by inclusion:

$$P_d/I_d \longrightarrow P_{d+1}/I_{d+1}$$

are all one to one. Since we assume the quadrature formula to be interpolatory, the evaluation map:

$$P_n/I_n \longrightarrow \mathbf{C}^N,\ \ p \mapsto (p^{(j)}(a_k))_{j,k},$$

is an isomorphism. Thus

$$dim(P_d/I_d) \le N,\ \ 0 \le d \le n,$$

with equality for $d = n$.

The following simple remark will be essential for the characterization of minimal quadrature formulas:

$$I_d \subset P_d \ominus FP_{n-d},\ \ 0 \le d \le n. \tag{19}$$

Indeed, if $p \in I_d$ and $q \in P_{n-d}$, then:

$$\langle p, Fq \rangle = \langle pq, \mathbf{1} \rangle = 0.$$

Thus we obtain a lower bound for the number N of nodes in the quadrature formula (18):

$$N \ge \max_{0 \le d \le n} dim[P_d/(P_d \ominus FP_{n-d})]. \tag{20}$$

Our next aim is to study the function:

$$\kappa_d = dim[P_d/(P_d \ominus FP_{n-d})].$$

Since $FP_0 = P_0 = \mathbf{C1}$ we find $\kappa_0 = \kappa_n = 1$.

Lemma 5.2. *Let d_0 be the largest integer $d \in [0, n]$ satisfying $P_d \ominus FP_{n-d} = 0$. Then $\kappa_{d_0+l} \le \kappa_{d_0} = d_0 + 1$, for all $l \in [0, n-d_0]$.*

Proof. Assume that $P_d \ominus FP_{n-d} \ne 0$ and $d < n$. Let a be an arbitrary complex number. Then the multiplication map by $z - a$ is well defined and injective:

$$P_d \ominus FP_{n-d} \xrightarrow{z-a} P_{d+1} \ominus FP_{n-d-1}.$$

Indeed, if $p \in P_d$ and $q \in P_{n-d-1}$, then

$$\langle (z-a)p, Fq \rangle = \langle p, (M^* - \overline{a})Fq \rangle = \langle p, F((z-a)q) \rangle = 0.$$

By choosing a to be distinct from the roots of a non-zero element of $P_{d+1} \ominus FP_{n-d-1}$, the map $z-a$ above will not be surjective.

Thus we have proved that

$$dim(P_{d+1} \ominus FP_{n-d-1}) > dim(P_d \ominus FP_{n-d}).$$

But $dim P_{d+1} = dim P_d$, so that $\kappa_{d+1} \leq \kappa_d$.

Let d_0 be as in the statement. Then $\kappa_{d_0} = d_0+1$ and $\kappa_{d_0+1} \leq d_0+2-1 = d_0+1$. □

The ideal I is principal, generated by a monic polynomial p of degree N:

$$p(z) = \prod_{k=1}^{m} (z-a_k)^{n_k}.$$

We know already that $d_0 < N \leq n$, and that $p \in P_d \ominus FP_d$ for all $d \geq N$. If we seek minimal interpolatory quadrature formulas, then $N = d_0+1$ and p is subject to the only restriction $p \in P_{d_0+1} \ominus FP_{n-d_0-1}$.

Indeed, with such a choice, the ideal $I = (p)$ satisfies $dim(P_n/I_n) = N$ and the integration functional on P_n factors through P_n/I_n: if $q \in P_{n-N}$, then

$$\langle pq, \mathbf{1} \rangle = \langle p, Fq \rangle = 0.$$

Therefore the existence of a quadrature formula with nodes, counting multiplicities, equal to the zeroes of p, is established.

In conclusion, we have proved the following result.

Theorem 5.3. *Let μ be a rapidly decreasing, positive measure on $\mathbf{C}$ and let F be the associated generalized Friedrichs operator on $P^2(\mu)$. Let n be a positive integer and let d_0 be the largest $d \in [0, n)$ with the property that $P_d \ominus FP_{n-d} = 0$.*

The set of all minimal interpolatory formulas of degree n of μ is in bijective correspondence with the monic polynomials $p \in P_{d_0+1} \ominus FP_{n-d_0-1}$.

For such a polynomial p, the corresponding quadrature formula has nodes equal to the roots of p.

Thus, the theorem above proves in particular the existence of minimal quadratures. For a fixed polynomial p, the corresponding weights in the quadrature formula can be obtained from the moments of degree n of μ, as in the classical case, by linear algebra. A simple dimension count shows that $dim[P_{d_0+1} \ominus FP_{n-d_0-1}] \leq 2$.

The case when the support of μ is real corresponds to classical Gaussian cubatures. Indeed, since $supp(\mu) \subset \mathbf{R}$ implies $Fp = p$ for all real polynomials p, the above method of finding minimal cubatures yields the next well known result.

Corollary 5.4. *Let μ be a rapidly decreasing measure supported by a subset of the real line and let n be a positive integer. Then:*

a). If $n = 2k+1$, then there exists a unique minimal cubature formula of degree n, whose nodes coincides with the (simple) zeroes of the orthogonal polynomial $p \in P_{k+1} \ominus P_k$;

b). If $n = 2k$, then the set of minimal cubature formulas of degree n is parametrized by all monic polynomials $p \in P_{k+1} \ominus P_{k-1}$.

In parallel with the theory of "mother bodies" of planar domains with real analytic boundary, see [25], we analyze below the following situation. Suppose, besides the measure μ above, a second positive measure $\hat{\mu}$, rapidly decaying at infinity is given, so that:

$$\int p d\mu = \int p d\hat{\mu}, \qquad p \in \mathbf{C}[z]. \tag{21}$$

As in the case of μ, we assume that the support of $\hat{\mu}$ is infinite.

The typical example for such a pair of measures is $\mu = \chi_E dA$ and $\hat{\mu} = const.\sqrt{1-x^2}dx$, where E is an ellipse with foci at ± 1, see [25].

Let P_d, $\hat{P}_d$ be the spaces of polynomials of degree less or equal than d, regarded as subspaces of $P^2(\mu)$, respectively $P^2(\hat{\mu})$. Let $F, \hat{F}$ be the corresponding Friedrichs operators. The following simple remark is all we need.

The identity map $id : P_d \longrightarrow \hat{P}_d$ is an isomorphism, and for every pair of non-negative integers k, l, we have:

$$id[P_k \ominus F P_l] = \hat{P}_k \ominus \hat{F} \hat{P}_l.$$

Indeed, if $p \in P_k$ and $q \in P_l$, then

$$\langle p, \hat{F} q \rangle_{P^2(\hat{\mu})} = \int p q d\hat{\mu} = \int p q d\mu = \langle p, F q \rangle_{P^2(\mu)}.$$

Thus we have in particular proved the following result.

Theorem 5.5. *Let $\mu, \hat{\mu}$ be a pair of measures with the same complex moments, both of infinite support.*

For every positive integer n the minimal interpolatory quadrature formulas of degree n, of μ and $\hat{\mu}$, coincide.

Moreover, the spaces $P_d \ominus F P_{n-d}$ and $\hat{P}_d \ominus \hat{F} \hat{P}_{n-d}$ which parametrize these quadratures, are identical.

For instance, in the case of the ellipse E, the minimal quadrature formulas are supported by the zeroes of the Chebyshev polynomials, i.e. orthogonal polynomials with respect to the measure $\sqrt{1-x^2}dx$, on the interval $[-1, +1]$. A remarkable feature of this example is that the polynomials in $P_d \ominus F P_{n-d}$ are the same for all confocal ellipses E.

The same phenomenon holds for a continuous family of equipotential domains satisfying a generalized quadrature identity given by a positive measure ν compactly supported on the real axis. To be more specific, let

$\Omega_t, 0 < t < t_0$, be the domains constructed by inverse balayge as in [27] and satisfying for all $t \in (0, t_0)$:

$$\int_{\Omega_t} f dA = e^t \int_{\mathbf{R}} f d\nu, \qquad f \in \mathbf{C}[z].$$

Then the orthogonal complement $P_d \ominus FP_{n-d}$, considered in the metric of the Hilbert space $L^2(\Omega_t, dA)$ does not depend of t. Consequently the nodes of the minimal quadrature formula are located on the real axis, and can be identified, in terms of the measure ν only, as in Corollary 5.4.

Finally we mention that the location of the zeroes of the polynomials $p \in P_{d_0+1} \ominus FP_{n-d_0-1}$, for a fixed measure μ as in Theorem 5.3, seems to be the most difficult part of this theory.

6. Friedrichs operators with symbol

In analogy to the theories of Hankel and Toeplitz operators, we introduce below a class of " Friedrichs operators with symbol". The simplest motivation for this generalization is the pull back of the standard Friedrichs operator by a conformal map.

If not otherwise stated, throughout this section Ω is an arbitrary planar domain. For a function $a \in L^\infty(\Omega)$ we denote by $M_a \in L(L^2(\Omega, dA))$ the multiplication operator $(M_a f)(z) = a(z)f(z),\ z \in \Omega,\ f \in L^2(\Omega)$. As before, P stands for the Bergman projection.

The *Friedrichs operator* F_a *with symbol* a is then the anti-linear operator :

$$F_a : AL^2(\Omega) \longrightarrow AL^2(\Omega), \qquad F_a = PM_aC.$$

It is clear that $\|F_a\| \le \|a\|_\infty$.

The next property appears implicitly in [17].

Lemma 6.1. *Let* $\phi : \Omega^\natural \longrightarrow \Omega$ *be a conformal map and let* $U : L^2(\Omega) \longrightarrow L^2(\Omega^\natural)$ *be the associated unitary map* $Uf = (f \circ \phi)\phi'$.

Then for every function $a \in L^\infty(\Omega)$ *the corresponding Friedrichs operators satisfy:*

$$F_a = U^* F_b^\natural U, \tag{22}$$

where $b = (a \circ \phi)\phi'/\overline{\phi'}$.

The proof is straightforward and it is left to the reader.

In analogy to the compactness criteria known for the original Friedrichs operator, see [7], [28], [30], [17], [25], we investigate below such a property for the operators F_a. The spirit of the proofs below is that of [30].

Theorem 6.2. *Let* Ω *be a bounded planar domain with boundary* Γ *and suppose that* $a \in C(\overline{\Omega})$. *If for each point* $\zeta \in \Gamma$ *(at least) one of the following assertions is true:*

(i) For some $\epsilon > 0$, *the set* $\Gamma \cap \{z;\ |\zeta - z| < \epsilon\}$ *is a* $C^{1+\alpha}$ *Jordan arc, for some* $\alpha > 0$;

or,

(ii) $a(\zeta) = 0$,

then the operator F_a *is compact.*

Proof. As in [30], we go back to the definition of a compact operator and prove that under the stated conditions, if (f_j) is a sequence in $AL^2(\Omega)$ which converges weakly to zero, then the sequence $\|Ff_j\|$ tends to zero.

For $f, g \in L^2(\Omega)$ we have:

$$\langle F_a f, g\rangle = \langle M_a \overline{f}, g\rangle = \langle a, fg\rangle,$$

therefore

$$\|F_a f\| = \sup_{\|g\|=1} |\int \overline{a} fg dA|.$$

We have to prove that

$$\lim_{j\to\infty} \sup_{\|g\|=1} |\int \overline{a} f_j g dA| = 0.$$

Assume by contradiction that this is false. Hence there exists a constant $c > 0$ and a sequence $g_j \in AL^2(\Omega)$, $\|g_j\| = 1$, such that:

$$|\int \overline{a} f_j g_j dA| \geq c, \quad j \geq 1. \tag{23}$$

Write $h_j = f_j g_j$ so that h_j is a bounded sequence in $AL^1(\Omega)$, and $h_j(z) \to 0$ for every point $z \in \Omega$. By theorems contained in Chapter 8 of [30], there exists a subsequence, still denoted by h_j, such that the measures $h_j dA$ converge weak* to a signed measure ν carried on Γ_{irr}, that is on the set of those points $\zeta \in \Gamma$ where condition (i) does not hold.

Thus we find that $|\int \overline{a} d\nu| \geq c$, but $\overline{a}$ vanishes on $supp(\nu)$, a contradiction. □

Thus, in studying the compactness behaviour of the operator F_a, one can localize the symbol a at points of Γ. As a direct application of the above proof we state the following result.

Corollary 6.3. *Let* $a, b \in C(\overline{\Omega})$. *If the operator* F_a *is compact, so is* F_{ab}.

As Friedrichs showed, if $\partial\Omega$ is smooth except for one "corner" where there is an angle $\alpha \neq \pi, 2\pi$, then the operator $F = F_1$ cannot be compact. He did this by showing that the self-adjoint operator $S = F^2$ has in its essential spectrum the number $(\frac{\sin\alpha}{\alpha})^2$. Here we outline another, computationally perhaps simpler way to establish non-compactness in this situation, that is also applicable with slight modifications to the boundary Friedrichs operator, to be discussed in the next section.

Proposition 6.4. *Assume that the boundary of the bounded domain* Ω *contains a corner of angle* $\alpha \neq \pi, 2\pi$. *Then the operator* F *is not compact.*

Proof. We will assume for simplification, although not necessary, that the boundary has only one corner z_0. Let $\rho > 0$ be a sufficiently small radius so that the set $\Omega \cap B(z_0, \rho)$ has a simply connected component Ω_0 having z_0 on its boundary.

Let ψ be a non-negative C^∞ function on $\mathbf{C}$ supported by a compact subset of the ball $B(z_0, \rho)$ and satisfying $\psi = 1$ on $B(z_0, \rho/2)$. Denote $a = \psi\chi_\Omega$. Then $F = F_a + F_{1-a}$; by the above theorem the operator F_{1-a} is compact since its symbol vanishes at the irregular point of $\partial\Omega$. Thus it suffices to show that F_a is not compact.

Next, map the wedge $W = \{w : |argw| < \alpha/2\}$ by a conformal map $z = \psi(w)$ on Ω_0, so that $\psi(0) = z_0$. According to Lemma 6.1, the operator F_a is unitarily equivalent to the Friedrichs operator on W with symbol $b = (a \circ \phi)(\phi'/\overline{\phi'})$. It is easy to see that ϕ' is continuous on $\overline{W}$ and non-vanishing near $w = 0$.

Thus our assertion is reduced to proving that if the function $b \in C(\overline{W})$ is bounded, with $b(0) \neq 0$, then the operator F_b on W is not compact.

Let $f \in AL^2(W)$ be a function of order $O(|w|^{-2})$ near ∞ and satisfying $\int_\Omega f^2 dA \neq 0$. This is possible because $\alpha \neq \pi$ and hence W is not a "null quadrature domain". For instance $f(w) = (w+1)^{-3}$ is such a function.

Then the functions $f_n(w) = nf(nw),\ n \geq 1$, have equal L^2 norms on W and tend weakly to zero in $AL^2(W)$.

But $F_b f_n$ do not tend in norm to zero. Indeed,

$$\langle f_n, F_b f_n \rangle = \int_W \bar{b} f_n^2 dA =$$

$$\int_W \overline{b(w)} n^2 f(nw)^2 dA = \int_W \overline{b(w/n)} f(w)^2 dA \to \overline{b(0)} \int_W f^2 dA \neq 0,$$

by Lebesgue's dominated convergence theorem. □

In view of Theorem 6.2 we obtain the next application.

Corollary 6.5. *In the conditions of Proposition 6.4, let $a \in C(\overline{\Omega})$. If $a(z_0) \neq 0$, then the operator F_a is not compact.*

Exactly as in the case of the Friedrichs operator, we note the following useful intertwining property. We denote as usual by $T_h = PM_h : AL^2(\Omega) \longrightarrow AL^2(\Omega)$ the *Toeplitz operator* with symbol h.

Lemma 6.6. *Let $a \in L^\infty(\Omega)$ and let $h \in H^\infty(\Omega)$. Then:*

$$T_h^* F_a = F_a T_h. \tag{24}$$

Proof. The proof is a simple chain of operator identitites:

$$T_h^* F_a = PM_{\overline{h}} PM_a CP = PM_{\overline{h}a} CP = PM_a CM_h P = F_a T_h.$$

□

By using this identity and following the proof in [25] one can show that for any symbol $a \in L^\infty(\Omega)$ the corresponding operator F_a cannot be Fredholm.

7. The boundary Friedrichs operator

Let Ω be a bounded domain with smooth boundary and let $H^2(\Omega)$ be the Hardy space, regarded as a closed subspace of $L^2(\Omega, ds)$; let P denote the corresponding orthogonal projection. To simplify the coming computations we asume that the measure ds is normalized and has mass one. By considering the complex conjugation operator $C : L^2(\Omega, ds) \longrightarrow L^2(\Omega, ds)$ a boundary Friedrichs operator $B = PC : H^2(\Omega) \longrightarrow H^2(\Omega)$ can therefore be defined in analogy to the Bergman space situation considered at the beginning of this note. The purpose of the present section is to outline a few relevant properties of this generalized Friedrichs operator.

First we consider the ideal situation of a domain bounded by a real analytic Jordan curve. Our aim is to draw a parallel to the classical Riesz-Herglotz representation in the unit disk, cf. for instance [6], and show that the boundary Friedrichs operator naturally appears in at least one such formula.

Let $S(z, \overline{w})$ be the Szegö kernel, that is the reproducing kernel of the Hardy space $H^2(\Omega)$, cf. [3]. Then any analytic function $f \in H^2(\Omega)$, assimilated with its non-tangential boundary values, satisfies:

$$f(z) = \int_{\partial\Omega} f(\zeta) S(z, \overline{\zeta}) ds(\zeta), \qquad z \in \Omega.$$

Similarly we have:

$$Bf(z) = \int_{\partial\Omega} \overline{f(\zeta)} S(z, \overline{\zeta}) ds(\zeta), \qquad z \in \Omega.$$

By denoting $u = \Re f$ and adding the above equations we obtain:

$$\frac{1}{2}(f + Bf)(z) = \int_{\partial\Omega} u(\zeta) S(z, \overline{\zeta}) ds(\zeta), \qquad z \in \Omega. \tag{25}$$

An equally simple argument leads to the following companion formula:

$$\frac{1}{2}(f(z) + \overline{f(w)}) S(z, \overline{w}) = \int_{\partial\Omega} u(\zeta) S(\zeta, \overline{w}) S(z, \overline{\zeta}) ds(\zeta), \qquad z, w \in \Omega. \tag{26}$$

Both the above representations can be interpreted as generalizations of the classical Riesz-Herglotz formula in the disk. The first one is of interest for instance when the operator B has finite rank, in which case a solution of the Dirichlet problem for the Laplace operator can be deduced from it by linear algebra in a finite dimensional space.

Also, under the generous conditions imposed on Ω we will see below that 1 is an isolated point in the spectrum of B^2, so that the operator $(I + B)$ can be inverted on the subspace $H^2_0(\Omega)$ of functions f having zero mean on $\partial\Omega$: $\langle f, \mathbf{1} \rangle = 0$. This yields an arbitrary function $f(z)$ as an integral operator applied to $u(\zeta)$. To be more precise, let $a = \langle f, \mathbf{1} \rangle$, so that the function $f - a\mathbf{1}$ has zero mean on the boundary and $\langle u, \mathbf{1} \rangle = \Re a$. By subtracting formula (25) written for the constant function a from that of f we obtain a Riesz-Herglotz type formula.

Lemma 7.1. *Let Ω be a simply connected domain with real analytic boundary and let $f \in H^2(\Omega)$. Then*

$$f = \langle f, \mathbf{1} \rangle \mathbf{1} + (\frac{I+B}{2})^{-1} \int_{\partial\Omega} (S(*, \overline{\zeta}) - \mathbf{1}) \Re f(\zeta) ds(\zeta). \tag{27}$$

Note that the inverse of $(I+B)$ acts on an element of $H^2_0(\Omega)$ and that $\Re a = \Re\langle f, \mathbf{1}\rangle$ is determined by the boundary values $u(\zeta) = \Re f(\zeta),\ \zeta \in \partial\Omega$, while $\Im a = \Im\langle f, \mathbf{1}\rangle$ remains, as expected, free. We leave it to the interested reader to develop the ramifications of this formula.

The second formula, (26), contains a more symmetric, positive definite Poisson type kernel $P(z, \zeta)$ of the domain Ω: $P(z,\zeta) = \frac{|S(z,\overline{\zeta})|^2}{S(z,\overline{z})}$; see [3] for more details of this nature. In the (only) well studied case of a disk (or polydisk), formula (26) is the link between the Riesz-Herglotz representation and classical interpolation problems, see [15].

Returning to formula (25), a conformal mapping argument, or simply the inspection of the regularity of the integral kernel of the Friedrichs operator, shows that B extends uniquely (by continuity) to any function $f \in H^1(\Omega)$, where now $Bf \in H^p(\Omega), p < 1$, and therefore it has almost everywhere boundary values. We will denote the extension by the same symbol Bf. This immediately leads to the following representation formula, also of classical flavour.

Proposition 7.2. *Let Ω be an analytic, smooth, Jordan domain and let f be an analytic function in Ω satisfying $\Re f > 0$ in Ω. Let μ be the positive measure supported by $\partial\Omega$ obtained as distributional boundary values of $\Re f(\zeta)$. Then*

$$\frac{1}{2}(f + Bf)(z) = \int_{\partial\Omega} S(z, \overline{\zeta}) d\mu(\zeta), \qquad z \in \Omega. \tag{28}$$

Conversely, any positive measure μ on $\partial\Omega$ produces by the above formula an analytic function f in Ω, unique up to an imaginary constant, and satisfying $\Re f > 0$ in Ω.

Proof. It is known that any positive measure μ on $\partial\Omega$ is the distributional boundary limit of a positive harmonic function $\Re f > 0$. To be more specific, this means that μ is the weak* limit of measures $\Re f ds$ carried on suitable curves bounding domains exhausting Ω, see [6].

We have only to prove the fact that, if $f \in H^1(\Omega)$ and $(I+B)f = 0$, then f is a purely imaginary constant. Let $h \in H^\infty(\Omega)$. By continuity we infer:

$$\int_{\partial\Omega} f h ds = - \int_{\partial\Omega} \overline{f} h ds.$$

But this shows that the harmonic function $\Re f$ is orthogonal to $H^\infty(\Omega) + CH^\infty(\Omega)$. The regularity assumption on the boundary, and the simply connectdness of Ω imply then that $\Re f = 0$, hence $f = iC$ with $C \in \mathbf{R}$. $\square$

From this point on we assume that Ω is a domain with rectifiable boundary Γ. The Szegö projector will be denoted by $P : L^2(\Gamma, ds) \longrightarrow H^2(\Gamma, ds)$, and for a function $a \in L^\infty(\Gamma)$ we define the *boundary Friedrichs operator with symbol* a as: $B_a = PM_aC$. In parallel to the case of the Bergman space we will be interested in the compactness of this operator.

Again, if $\phi : \Omega^\natural \longrightarrow \Omega$ is a conformal map, then the operator $B_a \in L(H^2(\Omega))$ is unitarily equivalent to the operator $B_b^\natural \in L(H^2(\Omega^\natural))$ with symbol $b = (a \circ \phi)(\phi'/\overline{\phi'})$. We omit the nearly identical details.

Theorem 7.3. *Let Ω be a bounded domain with C^1 smooth boundary Γ and let $a \in C(\Gamma)$. Then the operator B_a is compact.*

Proof. The proof follows the same scheme as in the Bergman space case: assume that (f_j) is a weakly convergent to zero sequence in $H^2(\Omega)$. We have to show that

$$\lim_{j\to\infty} \sup_{\|g\|_{2,\Gamma}=1} \left| \int_\Gamma \overline{a(z)} f_j(z) g(z) ds \right| = 0.$$

By our regularity assumption, on the boundary Γ one can write $dz = \tau ds$, where τ denotes the unit tangent vector at $z \in \Gamma$, and the function $\tau(z)$ is continuous. Thus we must prove that

$$\lim_{j\to\infty} \sup_{\|g\|_{2,\Gamma}=1} \left| \int_\Gamma \overline{a(z)\tau(z)} f_j(z) g(z) dz \right| = 0.$$

Now $h_j = f_j g$ are in $H^1(\Omega)$ with bounded norms and it is easy to check that $h_j(z) \to 0$ for each $z \in \Omega$. Since $a\tau \in C(\Gamma)$, the proof will be completed by the next proposition. □

Proposition 7.4. *Let (h_j) be a sequence in $H^1(\Omega)$ which is bounded and tends pointwise to zero in Ω. Then the measures $(h_j dz)$ on Γ tend to zero in the weak* topology.*

Proof. We are given that

$$h_j(\zeta) = \frac{1}{2\pi i} \int_\Gamma \frac{h_j(z)dz}{z - \zeta} \to 0, \qquad \zeta \in \Omega.$$

Moreover,

$$0 = \int_\Gamma \frac{h_j(z)dz}{z - \zeta}, \qquad \zeta \in \mathbf{C} \setminus \overline{\Omega}.$$

Hence

$$\lim_{j\to\infty} \int_\Gamma h_j(z) R(z) dz = 0,$$

for any rational function R with poles off Γ.

Since the area of Γ is zero by our regularity assumption, and the $L^1(\Gamma, ds)$ norms of h_j are bounded, the Hartogs-Rosenthal approximation theorem ([8]) allows us to replace R in the last equation by an arbitrary continuous function on Γ, which concludes the proof. □

Exactly as in the preceding section we can treat the situation of domains with finitely many corners and obtain the next result.

Proposition 7.5. *Let Ω be a domain with smooth boundary, except a corner of angle $\alpha \neq \pi, 2\pi$. Then the boundary Friedrichs operator B_1 is not compact.*

Finally, let us remark that Lemma 6.6 has a counterpart for the boundary Friedrichs operator: if $a \in L^\infty(\Gamma)$ and $h \in H^\infty(\Omega)$, then:

$$T_h^* B_a = B_a T_h,$$

where $T_h = PM_hP$ is the corresponding Toeplitz operator. Thus, as before, the operator B_a cannot be Fredholm.

8. Friedrichs operators on the unit disk

Of special interest is the case of the unit disk $\Omega = \mathbf{D}$. We denote by $z = e^{i\theta}$ the coordinate on the unit circle $\mathbf{T}$, and work with the normalized arc length measure $ds = \frac{1}{2\pi} d\theta$. The standard orthonormal basis of $L^2 = L^2(\mathbf{T}, ds)$ is $(z^n)_{n\in\mathbf{Z}}$ and the vector z^0 will also be denoted $\mathbf{1}$. We recall that for a function $a \in L^\infty(\mathbf{T})$ the associated *Hankel operator* $H_a : H^2 \longrightarrow L^2 \ominus H^2$ is: $H_a = (I-P)M_aP$. Henceforth we regard H_a, by natural coextension, as an operator from H^2 to L^2.

Lemma 8.1. *Let $a \in L^\infty(\mathbf{T})$. The boundary Friedrichs operator B_a and the Hankel operator $H_{\overline{a}}$ are related by the following equation:*

$$B_a = CH_{\overline{a}} + \mathbf{1} \otimes a^*, \tag{29}$$

where $a^*(z) = \overline{a(\overline{z})}$.

Proof. It is enough to check the identity on a basis vector z^n, $n \in \mathbf{Z}$. Let $a(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ be the Fourier decomposition of the function a.
Then

$$B_a z^n = P\Big(\sum_{k=-\infty}^{\infty} a_k z^{k-n}\Big) = \sum_{k\geq n} a_k z^{k-n},$$

and on the other hand,

$$CH_{\overline{a}} z^n = C(I-P) \sum_{k=-\infty}^{\infty} \overline{a_k} z^{n-k} = \sum_{k>n} a_k z^{k-n}.$$

Thus the operator $B_a - CH_{\overline{a}}$ has rank one and can be identified with $\langle *, a^* \rangle \mathbf{1}$, that is the operator which maps z^n to $a_n \mathbf{1}$ for every $n \geq 0$.

□

Thus the twisted Hankel operator $CH_{\overline{a}}$ is a rank-one perturbation of B_a, and therefore if one is compact the other is, too. Now the compactness or Schatten-von Neumann norm estimates of Hankel operators on the unit disk is a well studied area, with important applications to function theory, operator theory, approximation theory and stochastic processes, see for references [4], [20], [22], [23], [24].

We present below as a continuation of the previous proofs a slightly different approach to the compactness of the Friedrichs operators B_a and F_a on the unit disk.

Theorem 8.2. *Let $a \in L^\infty(\mathbf{T})$. Consider the following conditions:*

(i) The boundary Friedrichs operator B_a is compact;

(ii) Every maximizing sequnce for the extremal problem (iii) is norm convergent;

(iii) The linear functional $\Lambda_a : H^1(\mathbf{T}) \longrightarrow \mathbf{C}$

$$\Lambda_a g = \frac{1}{2\pi i} \int_{\mathbf{T}} \overline{a} g dz,$$

attains a maximum on the unit sphere of H^1;

(iv) $\overline{a}$ has a unique closest element in H^∞.

(v) $\overline{a} \in H^\infty + C(\mathbf{T})$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) and (v) $\Leftrightarrow$ (i).

The equivalence between (i) and (v) is due to Nehari [20]; the implication from (iii) to (iv) is standard in the theory of dual problems, cf. [6] pp. 133; the implication from (iv) to (v) is not true as it was for instance remarked in [1].

We are grateful to one of the referees for the following couple of counterexamples. Quite specifically, let $f \in L^\infty = L^\infty(\mathbf{T})$ satisfy $dist(f, H^\infty + C) < dist(f, H^\infty)$. Then it is well known and it is easy to prove that f has a unique closest element in H^∞. However, under this condition f does not have to belong to $H^\infty + C$. To see that the implication $(iii) \Rightarrow (v)$ is false, consider a function $h \in H^2$ such that $f = \frac{h}{\overline{h}} \notin H^\infty + C$. It is easy to see then that the functional $L_{\overline{f}}$ attains a maximum at $g = h^2$.

In what follows we merely indicate the principal ideas in the proof of Theorem 8.2. As in the proofs contained in the last two sections, the main ingredient is the following lemma. The notations are unchanged.

Lemma 8.3. *A necessary and sufficient condition for the operator B_a to be compact is: for every sequence (h_n) in $H^1(\mathbf{T})$ with bounded norms such that $h_n(z) \to 0$ for all $z \in \mathbf{D}$ we have $\Lambda_a h_n \to 0$.*

Proof. Supose that B_a is compact, and let $h_n \in H^1$ be a sequence with $\|h_n\|_1 = 1$ and $h_n(z) \to 0$ pointwise in $\mathbf{D}$. One can factor $h_n = f_n g_n$ with $f_n, g_n \in H^2$ and $\|f_n\|_2 = \|g_n\|_2 = 1,\ n \geq 1$. By passing to subsequences we can assume that $f_n \to f$ and $g_n \to g$, in the weak topology of H^2. Then

$f_n g_n \to fg$ pointwise, hence $fg = 0$, that is $f = 0$ or $g = 0$. Without loss of generality we can assume $f = 0$. Since $\|B_a f_n\|_2 \to 0$, we conclude:

$$\Lambda_a h_n = \langle g_n, B_a f_n \rangle \to 0.$$

In the opposite direction, assuming $\Lambda_a h_n \to 0$ for every sequence $(h_n) \subset H^1$ converging weak* to zero, let f_n be a sequence of H^2 which converges weakly to zero. Then for every bounded sequence $(g_n) \subset H^2$ we have

$$\langle g_n, B_a f_n \rangle = \Lambda_a (f_n g_n) \to 0,$$

verifying the compactness of B_a. □

In the proof of $(i) \Rightarrow (ii)$ we need the following known result [21].

Lemma 8.4. *Let $h_n, h \in H^1$ satisfy $h_n \to h$ in the weak* topology and $\|h_n\| \to \|h\|$.*
Then $\|h_n - h\|_1 \to 0$.

Proof. We can assume $\|h_n\|_1 = \|h\|_1,\ n \geq 1$. Factor then $h_n = f_n g_n$ with $f_n, g_n \in H^2$, $\|f_n\|_2 = \|g_n\|_2 = 1$. By passing to subsequences we can assume that $f_n \to f$ and $g_n \to g$, weakly in H^2. Thus $1 \leq \|f\|_2 \|g\|_2 \leq 1$, therefore $\|f\|_2 = \|g\|_2 = 1$.
Then

$$\int_{\mathbf{T}} |h_n - h| ds = \int_{\mathbf{T}} |f_n g_n - f_n g + f_n g - fg| ds \leq$$

$$\int_{\mathbf{T}} (|f_n||g_n - g| + |f_n - f||g|) ds \leq \|f_n - f\|_2 + \|g_n - g\|_2.$$

But

$$\|f_n - f\|_2^2 = \|f_n\|_2^2 + \|f\|_2^2 - 2\Re \int_{\mathbf{T}} f \overline{f_n} ds \to 0,$$

and similarly $\|g_n - g\|_2 \to 0$.
In conclusion $\|h_n - h\|_1 \to 0$. □

Next we discuss the proof of $(i) \Rightarrow (ii)$ in Theorem 8.2. Suppose B_a compact and let

$$M = \|\Lambda_a\| = \sup_{h \in (H^1)_1} | \int_{\mathbf{T}} \overline{a} h ds |.$$

For a Banach space X, we have denoted above by X_1 its closed unit ball. Let (h_n) be a sequence in H^1, satisfying $\|h_n\|_1 = 1$ and $\Lambda_a h_n \to M$. We can assume that $M \neq 0$, otherwise the assertion is trivial. By passing to a subsequence we can assume that $h_n \to h$ pointwise, for some element $h \in H^1$. If we can show that $\|h - h_n\|_1 \to 0$ we are done since then $\Lambda_a h = M$.

Assume by contradiction, in view of the preceding lemma, that $\|h\|_1 < 1$. Then

$$\Lambda_a h = \Lambda_a h_n + \Lambda_a (h - h_n),$$

and the last integral tends to zero by Lemma 8.3. Thus $\Lambda_a h = M$ which contradicts $\|h\| < 1$.

The implication $(v) \Rightarrow (i)$ follows from Theorem 7.3, and the rest of the proof can be derived from the cited references: [1] or [20] for the implication $(i) \Rightarrow (v)$ and [6] for the implication $(iii) \Rightarrow (iv)$.

Theorem 8.2 and its proof show that the compactness of the boundary Friedrichs operator B_a is easier to characterize than its Bergman space counterpart. For the operator F_a a Nehari type condition is not yet known.

We can sum up what we know about the compactness of F_a for the unit disk as follows. We denote by

$$N(\mathbf{D}) = \{u \in L^\infty(\mathbf{D}) : \int_\Omega fudA = 0, \ f \in AL^1(\mathbf{D})\},$$

the annihilator of the space $AL^1(\mathbf{D})$.

Proposition 8.5. *Let $a \in L^\infty(\mathbf{D})$, and consider the assertions:*

(i) F_a is compact;

(ii) Any maximizing sequence for the extremal problem (iii) is norm convergent;

(iii) The linear functional

$$L_a : AL^1(\Omega) \longrightarrow \mathbf{C}, \qquad L_a(f) = \int_\Omega \overline{a} f dA,$$

attains a maximum on the unit sphere of $AL^1(\Omega)$;

(iv) $\overline{a}$ has a unique closest element in $N(\mathbf{D})$, in sup norm;

(v) $\overline{a}$ belongs to $C(\overline{\mathbf{D}}) + N(\mathbf{D})$.

Then $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ and $(v) \Rightarrow (i)$.

Remarks.

Observe that, since the annihilator space $N(\mathbf{D})$ is weak* closed in $L^\infty(\mathbf{D})$, every function in $L^\infty(\mathbf{D})$ does have at least one closest element in $N(\mathbf{D})$. For aspects of this see [14] and further references there.

Functions a in $L^\infty(\mathbf{D})$ for which the maximum of L_a is attained on the unit sphere of $AL^1(\mathbf{D})$ have a special interest in connection with an extremal problem in quasiconformal mapping, which translates into the question: for which a in $L^\infty(\mathbf{D})$ of norm one the functional L_a has norm one? In those cases where L_a attains a maximum this can only happen if a has the special form $|g|/g$ for some $g \in AL^1(\mathbf{D})$, and then a corresponds to the complex dilatation of a so-called Teichmüller mapping, see e.g. the references in [28].

Proof. (of Proposition 8.5.) (outline) The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are proved in a similar way to the corresponding steps in Theorem 8.2, based on the (known) analog of Lemma 8.4. The implication $(iii) \Rightarrow (iv)$ is standard approximation theory, see [14]. And $(v) \Rightarrow (i)$ has been done. ∎

References

[1] S. Axler, I. D. Berg, N. P. Jewell, A. Shields, *Approximation by compact operators and the space $H^\infty + C$*, Ann. Math. **109**(1979), 601- 612.

[2] M. B. Balk, *Polyanalytic Functions*, Akademie Verlag, Berlin 1991.

[3] S. Bergman, *The Kernel Function and Conformal Mapping*, Amer. Math. Soc., Providence, R. I., 1970.

[4] J. Bourgain, *On the similarity problem for polynomially bounded operators on Hilbert space*, Israel J. Math. **53**(1986), 315- 332.

[5] J. J. Carmona, K. Yu. Fedorovski, P. V. Paramonov, *On uniform approximation by polyanalytic polynomials and the Dirichlet problem for bianalytic functions*, preprint 1999.

[6] P. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.

[7] K. Friedrichs, *On certain inequalities and characteristic value problems for analytic functions and for functions of two variables*, Trans. Amer. Math. Soc. **41**(1937), 321-364.

[8] T. Gamelin, *Uniform Algebras*, Prentice Hall, Englewood Cliffs New Jersey, 1969.

[9] B. Gustafsson, *Quadrature identities and the Schottky double*, Acta Appl. Math. **1**(1983), 209-240.

[10] B. Gustafsson, *On mother bodies of convex polyhedra*, SIAM J. Math. Analysis **29**(1998), 1106 - 1117.

[11] V. Havin, *Approximation in the mean by analytic functions*, Dokl. Akad. Nauk. SSSR **178**(1968), 1025 - 1028.

[12] T. Kato, *Perturbation Theory for Linear Operators*, Springer Verlag, Berlin, 1995.

[13] D. Khavinson, J. McCarthy, H. S. Shapiro, *Best approximation in the mean by analytic and harmonic functions*, Indiana Univ. Math. J., to appear.

[14] D. Khavinson, H. S. Shapiro, *Best approximation in the supremum norm by analytic and harmonic functions*, Ark. för Mat., to appear.

[15] A. Korányi, L. Pukánsky, *Holomorphic functions with positive real part on polycylinders*, Trans. Amer. Math. Soc. **108** (1963), 449- 456.

[16] A. Lenard, *The numerical range of a pair of projections*, J. Funct. Analysis **10**(1972), 410-413.

[17] P. Lin , R. Rochberg, *On the Friedrichs operator*, Proc. Amer. Math. Soc. **123**(1995), 3335- 3342.

[18] D. Luecking, *The compact Hankel operators form an M-ideal in the space of Hankel operators*, Proc. Amer. Math. Soc. 79(1980), 222- 224.

[19] I. P. Mysovskikh, *Interpolatory Cubature Formulas* (in Russian), Nauka, Moscow, 1981.

[20] Z. Nehari, *On bounded bilinear forms*, Ann. Math. **65**(1957), 153-162.

[21] D. J. Newman, *Pseudo-uniform convexity in H^1*, Proc. Amer. Math. Soc. **14** (1963), 676- 679.

[22] N. K. Nikolskii, *Treatise on the Shift Operator*, Springer, Berlin, 1986.

[23] V. V. Peller, *Estimates of functions of power bounded operators on Hilbert spaces*, J. Operator Theory, **7**(1982), 341- 472.

[24] V. V. Peller, S.V. Hruscev, *Hankel operators, best approximations and stationary Gaussian processes*, Uspehi Math. Nauk **37**(1982), 53- 124.

[25] M. Putinar, H. S. Shapiro, *The Friedrichs operator of a planar domain* in S.A. Vinogradov Memorial Volume (V. Havin and N.K.Nikolskii, eds.), Birkhäuser Verlag, Basel, 2000, pp. 303-330.

[26] F. Riesz, B. Sz.-Nagy, *Functional Analysis*, Dover, New York, 1990.

[27] M. Sakai, *Quadrature Domains*, Lect. Notes Math. Vol. 934, Springer, Berlin, 1982.

[28] H. S. Shapiro, *Some inequalities for analytic functions integrable over a plane domain*, in Proc. Conf. "Approximation and Function Spaces" Gdansk 1979, North Holland, 1981, pp. 645- 666.

[29] H. S. Shapiro, *On some Fourier and distribution-theoretic methods in approximation theory*, in vol. *Approximation Theory. III*, Proc. Conf. held in Austin, Texas, 1980 (W. Cheney et al., eds.), Academic Press, San Diego, 1980, pp. 87- 124.

[30] H. S. Shapiro, *The Schwarz Function and its Generalization to Higher Dimensions*, Wiley-Interscience, New York, 1992.

[31] P. K. Suetin, *Polynomials Orthogonal over a Region and Bieberbach Polynomials*, Proc. Steklov Inst. Vol. **100**(1971), Amer. Math. Soc., Providence, R.I., 1974.

Department of Mathematics, University of California, Santa Barbara, CA 93106.

E-mail address: mputinar@math.ucsb.edu

Department of Mathematics, The Royal Institute of Technology, 10044 Stockholm, Sweden.

E-mail address: shapiro@math.kth.se