Polynomial optimization on odd-dimensional spheres

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Abstract. The sphere S^{2d-1} naturally embeds into the complex affine space $\mathbb{C}^d$. We show how the complex variables in $\mathbb{C}^d$ simplify the known Striktpositivstellensätze, when the supports are resticted to semi-algebraic subsets of odd dimensional spheres.

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1. Preliminaries

Let $\mathbb{C}^d$ denote complex Euclidean space with Euclidean norm given by $|z|^2 = \sum_{j=1}^{d} |z_j|^2$. The unit, odd dimensional sphere

$$S^{2d-1} = \{z \in \mathbb{C}^d;\ |z| = 1\}$$

is a particularly important example of a Cauchy-Riemann (usually abbreviated CR) manifold. This note will show how one can study problems of polynomial optimization over semi-algebraic subsets of S^{2d-1} by using the induced Cauchy-Riemann structure. Our results can be regarded as multivariate analogues of classical phenomena about positive trigonometric polynomials, known for a long time in dimension one ($d = 1$). They are also related to results concerning proper holomorphic mappings between balls in different dimensional complex Euclidean spaces and the geometry of holomorphic vector bundles.

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A polynomial map $p : \mathbb{C}^d \times \mathbb{C}^d \to \mathbb{C}$ is called *Hermitian symmetric* if

$$p(z, \overline{w}) = \overline{p(w, \overline{z})}$$

for all z and w. By polarization one can recover a Hermitian symmetric polynomial from its real values $p(z, \overline{z})$. We therefore work on the diagonal (where $w = z$) and let $\mathcal{H} \subset \mathbb{C}[z, \overline{z}]$ denote the space of Hermitian symmetric polynomials on $\mathbb{C}^d$. Note that $\mathcal{H}$ is a real algebra, naturally isomorphic to the polynomial algebra $\mathbb{R}[x, y]$, where $z = x + iy \in \mathbb{R}^d + i\mathbb{R}^d$. Henceforth we will freely identify a Hermitian symmetric polynomial $P(z, \overline{z})$ with its real form $p(x, y) = P(x + iy, x - iy)$.

We denote by $\Sigma^2\mathcal{H}$ the convex cone consisting of sums of squares of Hermitian polynomials. We denote by $\Sigma^2_h\mathcal{H}$ the convex cone consisting of polynomials which are squared norms of (holomorphic) polynomial mappings. Thus $R \in \Sigma^2_h\mathcal{H}$ if there exist polynomials $p_j \in \mathbb{C}[z]$ such that

$$R(z, \overline{z}) = \sum_{j=1}^{m} |p_j(z)|^2.$$

See [11] and [1] for various characterizations of $\Sigma^2_h\mathcal{H}$. We have the containment

$$\Sigma^2_h\mathcal{H} \subset \Sigma^2\mathcal{H},$$

simply because

$$|p|^2 = (\frac{p + \overline{p}}{2})^2 + (\frac{p - \overline{p}}{2i})^2 = u^2 + v^2,$$

where u and v are the real and imaginary parts of p. The containment is strict as illustrated by the following two examples.

Example a). In one variable we define a polynomial R by

$$R(z, \overline{z}) = (z + \overline{z})^2 = 4x^2.$$

It is evidently a square but not in $\Sigma^2_h\mathcal{H}$. Note that the zero set of an element in $\Sigma^2_h\mathcal{H}$ must be a complex variety and thus cannot be the imaginary axis.

Example b). In two variables we define $R(z, \overline{z}) = (|z_1|^2 - |z_2|^2)^2$. Again R lies in $\Sigma^2\mathcal{H}$ but not in $\Sigma^2_h\mathcal{H}$. Here one can observe that elements of $\Sigma^2_h\mathcal{H}$ must be plurisubharmonic but that R is not. In 3.3 we will show additionally that R cannot be written as a squared norm on the unit sphere.

In this paper we are primarily concerned with optimization on the sphere. We therefore first let $I = I(S^{2d-1})$ be the ideal of $\mathcal{H}$ consisting of all polynomials vanishing on S^{2d-1}. We then define

$$\mathcal{H}(S^{2d-1}) = \mathcal{H}/I,$$

and regard it as a space of polynomial functions defined on the sphere. As a matter of fact, each real-valued polynomial p has a representative in $\mathcal{H}(S^{2d-1})$, when p is regarded as a function on the sphere.

In analogy with the above notations we denote by $\Sigma^2\mathcal{H}(S^{2d-1})$ the convex cone consisting of sums of squares of Hermitian polynomials on the sphere. We denote by $\Sigma_h^2\mathcal{H}(S^{2d-1})$ the convex hull of Hermitian squares:

$$\Sigma_h^2\mathcal{H}(S^{2d-1}) = \mathrm{co}\{|p(z)|^2;\ p \in \mathbb{C}[z]\} \mod I.$$

Let us pause for a moment and recall a classical one-dimensional result which is guiding our investigation. We include its elementary proof for convenience.

Lemma 1.1 (Riesz-Fejér). *A non-negative trigonometric polynomial is the squared modulus of a trigonometric polynomial.*

Proof. Let $p(e^{i\theta}) = \sum_{-d}^{d} c_j e^{ij\theta}$ and assume that $p(e^{i\theta}) \geq 0, \quad \theta \in [0, 2\pi]$. Since p is real-valued $c_{-j} = \overline{c_j}$ for all j. We set $z = |z|e^{i\theta}$ and extend p to the rational function defined by $p(z) = \sum_{-d}^{d} c_j z^j$. It follows that $p(z) = \overline{p(1/\overline{z})}$; furthermore its zeros and poles are symmetrical (in the sense of Schwarz) with respect to the unit circle.

Write $z^d p(z) = q(z)$. Then q is a polynomial of degree $2d$ whose modulus $|q|$ satisfies $|q| = |p| = p$ on the unit circle. In view of the mentioned symmetry one finds

$$q(z) = cz^\nu \prod_j (z - \lambda_j)^2 \prod_k (z - \mu_k)(z - 1/\overline{\mu_k}),$$

where c is a constant, $|\lambda_j| = 1$ and $0 < |\mu_k| < 1$.

Evaluating on the circle and using $|\zeta^2| = |\zeta|^2$ we obtain

$$p(e^{i\theta}) = |p(e^{i\theta})| = |q(e^{i\theta}| =$$

$$|c| \prod_j |e^{i\theta} - \lambda_j|^2 \prod_k \frac{|e^{i\theta} - \mu_k|^2}{|\mu_k|^2},$$

and hence p is the squared modulus of a trigonometric polynomial. $\square$

This fundamental lemma has deeply influenced twentieth century functional analysis. For instance the Riesz-Fejér Lemma is equivalent to the spectral theorem for unitary operators; see [28].

When invoking duality, the above is not less interesting. It was in this form that Riesz-Fejér Lemma was first generalized to an arbitrary dimension.

Lemma 1.2. *Let $L \in \mathcal{H}(S^{2d-1})'$ be a linear functional which is non-negative on $\Sigma_h^2\mathcal{H}(S^{2d-1})$. Then L is represented by a positive Borel measure supported on the sphere.*

The proof has implicitly appeared in the works of Ito [16], Yoshino [31], Lubin [21] and Athavale [3], all dealing with subnormality criteria for commuting tuples of bounded linear operators. Without aiming at completeness, here is the main idea.

Proof. (Sketch) Let L be a non-negative functional on $\Sigma_h^2\mathcal{H}(S^{2d-1})$. Fix a polynomial $p \in \mathbb{C}[z]$ and consider the functional

$$f(r_1^2, ..., r_d^2) \mapsto L(f|p(z)|^2), \quad f \in \mathbb{R}[r_1^2, ..., r_d^2],$$

where $r_j^2 = |z_j|^2$. Since

$$1 - |z_j|^2 = \sum_{k \neq j} |z_k|^2,$$

$$L(\prod_j [(1 - r_j^2)^{n_j} r_j^{2m_j}]|p|^2) \geq 0, \quad n_j, m_j \geq 0.$$

By a classical result of Haviland, see for instance [2], there exists a positive Borel measure $\mu_{|p|^2}$ on the simplex Δ defined by

$$\Delta = \{(r_1^2, ..., r_d^2);\ r_1^2 + ... + r_d^2 = 1\},$$

with the property

$$L(f|p(z)|^2) = \int_\Delta f d\mu_{|p|^2}.$$

The total mass of $\mu_{|p|^2}$ is $L(|p|^2)$.

By polarization, one can define complex valued measures by

$$L(fp\overline{q}) = \int_\Delta f d\mu_{p\overline{q}}, \quad f \in \mathbb{R}[r_1^2, ..., r_d^2],\ p, q \in \mathbb{C}[z],$$

so that the sesqui-linear kernel $(p, q) \mapsto \mu_{p\overline{q}}$ is positive semi-definite.

In short, the functional L can be extended to the linear space of functions (on the sphere) of the form

$$F(r,z) = \sum_{|\alpha|\le n} c_\alpha(r) z^\alpha,$$

where $c_\alpha(r)$ are bounded, Borel measurable functions on the simplex Δ. The extended functional $\tilde{L}$ still satisfies

$$\tilde{L}(|F(r,z)|^2) \ge 0.$$

Next we pass to polar coordinates $z_j = r_j\omega_j,\ |\omega_j| = 1$ and remark that multiplication by ω_j satisfies the isometric condition

$$\tilde{L}(|\omega_j F(r,z)|^2) = \tilde{L}(|F(r,z)|^2).$$

Thus, we can further extend the functional $\tilde{L}$ to all polynomials in r and $\omega, \overline{\omega}$, so that

$$\tilde{L}(|\omega_j^{-1} F(r,z)|^2) = \tilde{L}(|F(r,z)|^2)$$

and

$$\tilde{L}(|p(r,\omega,\overline{\omega})|^2) \ge 0.$$

We refer to [31] or [29] for the details how this extension is constructed. By rewriting the latter positivity condition we have in particular

$$\tilde{L}(|h(z,\overline{z})|^2) \ge 0, \quad h \in \mathbb{C}[z,\overline{z}],$$

whence, by the Stone-Weierstrass Theorem and the Riesz Representation Theorem, the functional $\tilde{L}$ is represented by a positive Borel measure, supported on the sphere.

The representing measure is unique by the Stone-Weierstrass Theorem. $\square$

2. A Striktpositivstellensatz

We now turn to the basic question considered in this paper. We are given a finite set of real polynomials in $2d$ variables $p, q_1, ..., q_r$, or equivalently, Hermitian symmetric polynomials in d complex variables. We suppose that $p(z,\overline{z})$ is strictly positive on the subset of S^{2d-1} where each q_j is nonnegative. Can we write p as a weighted sum of squared norms with q_i as weights, as the real affine Striktpositivstellensatz (see for instance [22]) suggests? The answer is yes, and we can offer at least two different reasons why it is so.

Theorem 2.1. *Let $p, q_1, ..., q_r \in \mathbb{R}[x, y]$, where $x + iy = z \in \mathbb{C}^d$. If*

$$(|z| = 1,\ q_i(z, \overline{z}) \geq 0, 1 \leq i \leq r) \ \Rightarrow \ (p(z, \overline{z}) > 0),$$

then

$$p \in \Sigma_h^2 + q_1\Sigma_h^2 + ... + q_r\Sigma_h^2 + I(S^{2d-1}).$$

First we discuss the history of such Hermitian squares decompositions, in the case where there are no constraints. A Hermitian symmetric polynomial p is called bihomogeneous of degree (m, m) if

$$p(\lambda z, \overline{\lambda z}) = |\lambda|^{2m} p(z, \overline{z})$$

for all complex numbers λ and all $z \in \mathbb{C}^d$. The values of a bihomogeneous polynomial are determined by its values on the sphere. When p is bihomogeneous and strictly positive on the sphere, Quillen [27] proved that there is an integer k and a homogeneous polynomial vector-valued mapping $h(z)$ such that

$$|z|^{2k} p(z, \overline{z}) = |h(z)|^2.$$

This result was discovered independently by the first author and Catlin [6] in conjunction with the first author's work on proper mappings between balls in different dimensions. The proof in [6] uses the Bergman projection and some facts about compact operators, and it generalizes to provide an isometric imbedding theorem for certain holomorphic vector bundles [7].

It is worth noting that the integer k and the number of components of h can be arbitrarily large, even for polynomials p of total degree four in two variables. The result naturally fits into the phenomena encoded into the old or recent Positivestellensätze, see for instance [22]. For the specific case of Hermitian polynomials on spheres see [8] for considerable discussion and generalizations.

Using a process of bihomogenization, Catlin and the first author (see [6], [8] and [9]) proved that if p is arbitrary (not necessarily bihomogeneous) and strictly positive on the sphere, then p agrees with a squared norm on the sphere; in other words, $p \in \Sigma_h^2 + I(S^{2d-1})$. Thus Theorem 1 holds when there are no constraints. Our proof of Theorem 1 first considers the case of no constraints, but we approach this case in a completely different manner.

Strict positivity is required for these results. The polynomial $(|z_1|^2 - |z_2|^2)^2$ is bihomogeneous and nonnegative everywhere, but there is no element in Σ_h^2 agreeing with it on the sphere. See Example 3.3 below.

Proof. (of Theorem 1) Suppose first that no q_i's are present and assume by contradiction that $p \notin \Sigma_h^2$, all regarded as elements of $\mathcal{H}(S^{2d-1})$. Since the constant function 1 belongs to the algebraic interior of the convex cone $\mathcal{H}(S^{2d-1})$, the separation lemma due to Eidelheit-Kakutani [12, 17] provides a linear functional $L \in \mathcal{H}(S^{2d-1})'$, satisfying both $L(1) > 0$ and

$$L(p) \le 0 \le L|_{\Sigma_h^2}.$$

According to Lemma 2, there exists a positive Borel measure μ, supported on the unit sphere, which represents L. Therefore

$$0 \ge L(p) = \int p d\mu > 0,$$

a contradiction.

The proof of the general case is similar, with the difference that we have to prove that the support of the measure μ is contained in the non-negativity set defined by the functions q_i. To this aim, fix an index i, and remark that

$$\int q_i |p|^2 d\mu \ge 0$$

for all $p \in \mathbb{C}[z]$. Now, by the first case, every positive polynomial $P(z, \overline{z})$ is in the convex hull of the Hermitian squares, whence

$$\int q_i P(z, \overline{z}) d\mu \ge 0$$

whenever $P(z, \overline{z}) > 0$ on the sphere, that is whenever $P(z, \overline{z}) \ge 0$ on the sphere. In view of Stone-Weierstrass Theorem, every continuous functions f on the sphere can be uniformly approximated by real polynomials. In particular, we infer

$$\int q_i f^2 d\mu \ge 0, \quad f \in C(S^{2d-1}).$$

But this inequality holds only if the support of μ is contained in the non-negativity set $q_i(z, \overline{z}) \ge 0$. □

3. Examples

3.1. Optimization on the closed disk

The following simple example shows that Hermitian sums of squares do not suffice as positivity certificates on more general semi-algebraic sets. Specifically, let

$$P(z, \overline{z}) = 1 - \frac{4}{3}|z|^2 + a|z|^4,$$

with $\frac{1}{3} < a$. Note that

$$P(z,\overline{z}) = (1 - \frac{2}{3}|z|^2)^2 + (a - \frac{4}{9})|z|^4,$$

and hence $P \in \Sigma^2\mathcal{H}$ when $a \geq \frac{4}{9}$. Hence we assume $\frac{1}{3} < a < \frac{4}{9}$. The polynomial $1 - \frac{4}{3}t + at^2$ is decreasing for $0 < t < 1$ when $a < \frac{2}{3}$; therefore $|z| \leq 1$ implies $P(z,\overline{z}) \geq 1 + a - \frac{4}{3} > 0$.

On the other hand,

$$P \notin \Sigma_h^2 + (1 - |z|^2)\Sigma_h^2.$$

To see that P is not in this set, we apply the hereditary calculus. See [1] for details. We replace z with a contractive operator T and replace $\overline{z}$ with T^*. We follow the usual convention of putting all T^*'s to the left of the powers of T. If P were in this set, we would obtain

$$\|T\| \leq 1 \quad \Rightarrow \quad p(T,\overline{T}) \geq 0.$$

In particular let T be the 2×2 Jordan block with 1 above the diagonal. We obtain a contradiction by computing that $P(T,T^*)$ is the diagonal matrix with eigenvalues 1 and $-\frac{1}{3}$.

On the other hand, the larger convex cone $\Sigma^2 + (1 - |z|^2)\Sigma_h^2$ is appropriate in this case, see [24, 26].

3.2. Squared norms

Recall that $\Sigma_h^2\mathcal{H}$ denotes the convex cone consisting of polynomials which are squared norms of (holomorphic) polynomial mappings. In all dimensions the zero set of an element in $\Sigma_h^2\mathcal{H}$ must be a complex variety.

Suppose $R(z,\overline{z}) \geq 0$ for all z. Even in one dimension we cannot conclude that $R \in \Sigma_h^2\mathcal{H}$. We noted earlier, where $x = \mathrm{Re}(z)$, the example

$$R(z,\overline{z}) = (z + \overline{z})^2 = 4x^2.$$

The zero set of R is the imaginary axis, which has no complex structure. In one dimension of course, the zero set of an element in $\Sigma_h^2\mathcal{H}$ must be either all of $\mathbb{C}$ or a finite set.

Things are more complicated and interesting in higher dimensions. Consider the following example from [8]. Define a Hermitian bihomogeneous polynomial in three variables by

$$p(z,\overline{z}) = (|z_1 z_2|^2 - |z_3|^4)^2 + |z_1|^8.$$

This polynomial p is nonnegative for all z, and its zero set is the complex plane given by $z_1 = z_3 = 0$ with z_2 arbitrary. Yet p is not a sum of squared moduli; even more striking is that p cannot be written as the quotient $\frac{|a(z)|^2}{|b(z)|^2}$ where a and b are sums of squared moduli. See [10] for additional information on this example and several tests for deciding whether a non-negative polynomial R can be written as a quotient of squared norms. See [30] for a necessary and sufficient condition involving the zeroes of R.

We give an additional example in one dimension. Define p by

$$p(z,\overline{z}) = 1 + bz^2 + \overline{b}\overline{z}^2 + c|z|^2 + |z|^4.$$

The condition for being a quotient of squared norms is that one of the following three statements holds:

$$c > 2|b|^2 - 2,$$

$$b = 0, c > -2,$$

$$|b| = 1, c = 0.$$

The condition for being nonnegative is simpler: $c \geq 2|b| - 2$.

3.3. Proof of Example b).

We claimed earlier that the polynomial $(|z_1|^2 - |z_2|^2)^2$ is bihomogeneous and non-negative everywhere, but there is no element in Σ_h^2 agreeing with it on the sphere.

Proof. Put $R(z,\overline{z}) = (|z_1|^2 - |z_2|^2)^2$, and let $V(R)$ denote its zero set. We note that $V(R) \cap S^{2n-1}$ is the torus T defined by $|z_1|^2 = |z_2|^2 = \frac{1}{2}$. Suppose for some polynomial mapping $z \to P(z)$ we have $R = |P|^2$ on the unit sphere. Note first that the zero set of $|P|^2$ is a complex variety. We have $|P(z)|^2 = 0$ for $z \in T$. We claim that P is identically zero. For each fixed z_2 with $|z_2| = 1$, the vector-valued polynomial mapping $z_1 \to P(z_1, z_2)$ vanishes on the circle $|z_1|^2 = \frac{1}{2}$ and hence vanishes identically. Since z_2 was an arbitrary point with $|z_2|^2 = \frac{1}{2}$ we conclude that the mapping $(z_1, z_2) \to P(z_1, z_2)$ vanishes whenever $z_1 \in \mathbf{C}$ and z_2 lies on a circle. By symmetry it also vanishes with the roles of the variables switched. It follows that the zero set of P (which is a complex variety) is at least three real dimensions, and hence P vanishes identically. Since R does not vanish identically on the sphere we obtain a contradiction. $\square$

3.4. Example

There exist non-negative polynomials R such that R is not in $\Sigma_h^2\mathcal{H}$, yet there is a positive integer N for which $R^N \in \Sigma_h^2\mathcal{H}$. The bihomogeneous polynomial R_λ given by

$$R_\lambda(z,\overline{z}) = (|z_1|^2 + |z_2|^2)^4 - \lambda|z_1z_2|^4$$

satisfies this property whenever $\lambda < 8$. See [30]. For $\lambda < 16$, $R_\lambda > 0$ on the sphere. By Theorem 1 it agrees with a squared norm *on the sphere*.

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