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ON A DIAGONAL PADÉ APPROXIMATION IN TWO COMPLEX VARIABLES

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ABSTRACT. A special type diagonal Padé approximation for a class of hermitian power series in two variables is related to a canonical strong-operator topology, finite-rank approximation of cyclic operators. The expected convergence of the process (uniform or in measure) is derived from operator theory facts.

1. INTRODUCTION

A reconstruction algorithm of planar shapes from their moments [15] has raised the approximation question which makes the subject of the present note. Although the situation in [15] is rather special, we consider below a more general setting, and in this new framework we prove some uniform convergence and convergence in measure results. Besides well known algebraic and analytic aspects of *one dimensional* Padé approximation, cf. for instance [2] and [27], we rely on von Neumann's theory of spectral sets, some basic properties of compact operators and the theory of hyponormal operators.

Recently, the algebra, convergence and algorithmic aspects of various Padé approximation schemes in several variables have received a lot of attention, see [7], [5] and the excellent survey [6]. The particular question treated in this note can be classified as an inhomogeneous, equation lattice approach to the Padeé approximation of a structured class of power series in two variables. For this problem the recent general results (such as those of [5], [7]) certainly apply. However, our framework is more particular and special from several points of view: the Hilbert space factorizations of series in two variables we deal with below reduce the Padé approximation technique to one variable (hence the simplicity of the proofs); due to the combination of operator theory techniques and approximation theory methods we obtain non-circular (in general bigger than expected) convergence domains (such as in Theorem 2.5 or Corollary 2.6); the linear algebra beyond the algorithmic part of the approximation scheme contained in this note is elementary and resonant to linear algebra aspects of orthogonal polynomials or numerical quadratures; some related numerical experiments are commented in [13] and [15].

The interplay between functions and operators we use in this paper is a variation on the idea of realization theory. A few similar frameworks are by now classical: [1], [11], [22]. The purpose of this note is to show that, even for the class of all linear operators, the realization theory link can produce interesting approximation theory results.

Next we discuss in more detail the contents of the note. The main problem, equivalent to the convergence of the diagonal Padé table, turns out to be: how the strong operator topology convergence of a sequence $T_n \to T$ of operators is trasmitted to the local resolvents $(T_d-z)^{-1}\xi \longrightarrow (T-z)^{-1}\xi$, beyond the boundary

[1]Paper partially supported by the National Science Foundation Grant DMS- 9800666

$|z| = \|T\|$? Classical results, such as Markov's theorem on the uniform convergence of the Padé approximants of the Cauchy transform of a positive measure, compactly supported on the real line, cf. [23] are contained in this setting.

Throughout this note, the object of approximation is a double series:

$$F(z,\overline{w}) = \sum_{m,n=0}^{\infty} \frac{b_{mn}}{z^{m+1}\overline{w}^{n+1}}, \tag{1}$$

whose coefficients satisfy, as a kernel, the positivity condition:

$$(R^2 b_{m,n} \pm b_{m+1,n+1})_{m,n=0}^{\infty} \geq 0, \tag{2}$$

where $R > 0$ is a constant. Under this assumption, the series F will be convergent in the polydomain $[\tilde{\mathbf{C}} \setminus \overline{D(0,R)}]^2$, where $\tilde{\mathbf{C}}$ is the one point compactification of the complex plane and $D(0,R)$ is the disk centered at 0, of radius R.

Standard factorization techniques, as for instance developed in the Appendix of [25], show then that there exists a unique linear bounded operator T, acting on a Hilbert space H, and a cyclic vecctor ξ of T, so that $\|T\| \leq R$ and :

$$F(z,\overline{w}) = \langle (T-z)^{-1}\xi, (T-w)^{-1}\xi \rangle; \qquad |z|, |w| > R. \tag{3}$$

Specifically, there exists a bijection between such series F and pairs (T,ξ), modulo joint unitary equivalence. Relation (2) can equivalently be written on Taylor coefficients as :

$$b_{mn} = \langle T^m \xi, T^n \xi \rangle, \qquad m,n \geq 0. \tag{4}$$

Thus, our problem is similar to a polarized, two-variable version of the well understood approximation by rational functions of transfer functions in linear control theory, see for instance [11] and the references cited there. Contrary to these studies, which put emphasis on the unitary dilation of a contractive operator T and then on the analysis of a continued fraction decomposition (known as the Schur algorithm) of the associated characteristic function, below we focus directly on a cyclic operator T and its localized resolvent $(T-z)^{-1}\xi$. Even this apparently less structured approach will prove to have certain advantages.

Assume that the Hilbert space $H = \bigvee_{n=0}^{\infty} T^n \xi$ has finite dimension equal to d, or equivalently that the $(d+1) \times (d+1)$ Gramm matrix $(b_{mn})_{m,n=0}^{d}$ is degenerate, of rank d. Then the function F is rational, of the form:

$$F(z,\overline{w}) = \frac{Q(z,\overline{w})}{P(z)\overline{P(w)}}, \tag{5}$$

where $Q(z,\overline{w})$ is a polynomial of degree $d-1$ in each variable, and $P(z)$ is a monic polynomial of degree d. Moreover, in this case a unique additive structure of Q is known (cf. [17]):

$$Q(z,\overline{w}) = \sum_{k=0}^{d-1} c_k q_k(z) \overline{q_k(w)},$$

where q_k are monic polynomails of a single complex variable, of degree k, and $c_k > 0,\ 0 \leq k \leq d-1$. The realization problem (which rational functions $F(z,\overline{z})$ arise as norms of local resolvents of matrices T) is also discussed in [17].

The interest for these functions lies in the fact that the "generalized lemniscates":

$$\Omega = \{z \in \mathbf{C};\ \frac{Q(z,\overline{z})}{P(z)\overline{P(z)}} \geq const.\},$$

approximate every planar domain, and for special pairs of polynomials P, Q, the above series F contains in finite form all the moments of Ω, see [15]. Then solving the moment problem is equivalent, exactly as in the one variable case [23], to approximating the series F by a sequence of naturally chosen Padé quotients.

In our situations we will select the diagonal approximants $F_d(z,\overline{w})$ of the form (5), which corresponds in the Hilbert space picture to taking the orthogonal projection π_d of H onto $\bigvee_{k=0}^{d-1} T^k\xi$. Specifically, we will show that:

$$F_d(z,\overline{w}) = \langle(\pi_d T\pi_d - z)^{-1}\xi, (\pi_d T\pi_d - w)^{-1}\xi\rangle; \quad |z|, |w| > R.$$

To put everything in one sentence, the approximation $F_n \to F$ will reflect, under various additional conditions, the fact that $so - \lim_{d\to\infty} \pi_d T\pi_d = T$ and $so - \lim_{d\to\infty} \pi_d T^*\pi_d = T^*$.

The contents is the following. Section 2 contains the approximation scheme and the proofs of its uniform convergence ouside the close numerical range of the operator T, respectively ouside the closed numerical range of a compact perturbation $T+K$, union possibly with a discrete set. Theorem 2.9 shows how general convergence in measure arguments in the theory of the univariate Padé table can be adapted to our Hilbert space framework. Section 3 contains a few examples of numerical series, obtained as double Cauchy transforms of various measures in $\mathbf{C}$, and the corresponding convergence results. Section 4 is a return to the original problem of shape reconstruction from moments, [15]. Finally, in Section 5 we investigate the stability of the proposed approximation algorithm in terms of the Taylor coefficients of the original function F.

The author thanks the referee for valuable bibliographical references.

2. The general approximation scheme

This section contains a review of known facts about the interplay between diagonal Padé approximation and finite-rank truncations of Hilbert space operators, see for instance [2]. Then some convergence results are derived from this connection.

Given a subset $\Lambda \subset \mathbf{Z}^2$, the notation $F(z,\overline{w}) \equiv 0 \;\; mod(\Lambda)$ means that the series F does contain only monomials $z^m\overline{w}^n$ with $(m,n) \in \Lambda$.

For every positive integer d we define the set:

$$\Lambda_d = \{(-m,-n);\; \max(m,n) > d+1, \; or \;\; m = n = d+1\}.$$

Exactly as in the classical theory of the Padé table, or its multidimensional analogues, see [2], [3], [6], [30], we start with the following simple algebraic fact.

Proposition 2.1. *Assume, with $d \geq 1$ fixed, that the coefficients of the series (1) satisfy:*

$$\det(b_{mn})_{m,n=0}^{d} \neq 0. \tag{6}$$

Then there exists a unique pair of polynomials $p_d(z), q_{d-1}(z,\overline{w})$, with the properties that $\deg(p_d) = d$, p_d is monic, q_{d-1} is of degree $d-1$ in each variable, and:

$$F(z,\overline{w}) - \frac{q_{d-1}(z,\overline{w})}{p(z)\overline{p(w)}} \equiv 0 \quad mod(\Lambda_d), \tag{7}$$

Note that a naive counting of real parameters in the complex symmetric matrix b_{mn} indexed over $(m,n), 0 \leq m,n \leq d, m+n < 2d$, and respecively the free coefficients in p_d and q_{d-1} leads to equality: $(d+1)^2 - 1 = d^2 + 2d$.

Proof. Condition (7) is equivalent to the fact that the Laurent series $p_d(z)\overline{p_d(w)}F(z,\overline{w}) - q_{d-1}(z,\overline{w})$ possibly contains monomials of the form $z^{-1}\overline{w}^{-1}$, or $z^{-m-1}\overline{w}^{-n-1}$, $\max(m,n) \geq 1$.

Let us write $p_d(z) = z^d + c_{d-1}z^{d-1} + \ldots + c_0$, with $c_j \in \mathbf{C}$, and $q_{d-1}(z,\overline{w}) = \sum_{k,l=0}^{d-1} a_{kl} z^k \overline{w}^l$, with $a_{kl} = \overline{a_{lk}} \in \mathbf{C}$. We also put $c_d = 1$ for the consistency of the coming formulae. Written on coefficients, condition (7) becomes:

$$a_{kl} = \sum_{i \geq k+1, j \geq l+1} c_i \overline{c_j} b_{i-1-k, j-1-l}, \qquad 0 \leq k, l \leq d-1, \tag{8}$$

and, for $k = -1$ and $j \geq l+1$:

$$0 = \sum_{i,j} c_i b_{i,j-l-1} \overline{c_j}.$$

A descending induction in l leads then to :

$$\sum_{i=0}^{d} c_i b_{ij} = 0, \qquad 0 \leq j < d. \tag{9}$$

Turning now to the Hilbert space factorization (4) of the coefficient matrix b_{mn} we obtain from (9):

$$\langle p_d(T)\xi, T^j\xi \rangle = 0, \qquad 0 \leq j < d. \tag{10}$$

Since the Gramm matrix of the vectors $\xi, T\xi, \ldots, T^d\xi$ was supposed to be nondegenerate, there exists exactly one polynomial p_d fulfilling the orthogonality conditions (10).

Once the polynomial p_d found, the coefficient matrix a_{kl} of the polynomial q_{d-1} is determined by the relations (8). This shows the existence and uniqueness of the pair (p_d, q_{d-1}) as in the statement. $\square$

The Hilbert space proof of Proposition 2.1 reduces the search of the polynomial p_d to a variational problem for a quadratic form, as follows.

Corollary 2.2. *With the notation above, the coefficients of the polynomial* $p_d(z) = c_d z^d + c_{d-1}z^{d-1} + \ldots + c_0$, $c_d = 1$, *satisfy:*

$$\min \sum_{i,j=0}^{d} \gamma_i b_{ij} \overline{\gamma_j} = \sum_{i,j=0}^{d} c_i b_{ij} \overline{c_j}, \tag{11}$$

where the minimum is taken among all systems of complex numbers $\gamma_0, \gamma_1, \ldots, \gamma_{d-1}$; $\gamma_d = 1$.

As it is expected, and known to experts, if the determinant (6) is 0, then a solution of the congruence (7) still exists, but it may not be unique.

Let us introduce a few new notations. Assume that the series (1) is given with the factorization (3), and fix a positive integer d. We denote $H_d = \bigvee_{k=0}^{d-1} T^k\xi$ and π_d will be the orthogonal projection of H onto H_d. The compressed operator $T_d = \pi_d T \pi_d$ will be regarded also as a linear transformation of H_d into itself.

Proposition 2.3. *With the above notation, the approximant in Proposition 2.1 is:*

$$\frac{q_{d-1}(z,\overline{w})}{p_d(z)\overline{p_d(w)}} = \langle (T_d - z)^{-1}\xi, (T_d - w)^{-1}\xi \rangle, \tag{12}$$

where p_d is the minimal polynomial of the operator $T_d \in L(H_d)$.

Proof. It suffices to remark that

$$T_d^k \xi = T^k \xi, \quad k < d, \tag{13}$$

and:

$$T_d^d \xi = \pi_d T^d \xi = \pi_d p_d(T)\xi - \pi_d(p_d(T) - T^d) =$$
$$(T^d - p_d(T))\xi = (T_d^d - p_d(T_d))\xi,$$

whence $p_d(T_d)\xi = 0$. Since the operator T_d is cyclic and of rank d (by the determinant condition (6)), we conclude that p_d is the minimal polynomial of T_d.

Since $p_d(z)\overline{p_d(w)}\langle (T_d - z)^{-1}\xi, (T_d - w)^{-1}\xi\rangle$ is a polynomial in both variables, relations (7) and (13) yield:

$$q_{d-1}(z, \overline{w}) = p_d(z)\overline{p_d(w)}\langle (T_d - z)^{-1}\xi, (T_d - w)^{-1}\xi\rangle,$$

and the proof is complete. □

Throughout this paper, the above canonical rational approximants will be denoted:

$$F_d(z, \overline{w}) = \langle (T_d - z)^{-1}\xi, (T_d - w)^{-1}\xi\rangle = \frac{q_{d-1}(z, \overline{w})}{p_d(z)\overline{p_d(w)}}.$$

We record below a few simple consequences of Proposition 2.3. The spectrum of a linear operator A will be denoted by $\sigma(A)$, while its numerical range will be $W(A) = \{\langle Ax, x\rangle; \ \|x\| = 1\}$. Recall that $\sigma(T) \subset \overline{W(T)}$ and that, by a theorem of Hausdorff, $W(T)$ is a convex set.

Corollary 2.4. *In the above conditions, we have:*

a). $\|T_d\| \leq \|T\|$; $so-lim_{d\to\infty} T_d = T,\ so-lim_{d\to\infty} T_d^* = T^*$.

b). The operator $T_d \in L(H_d)$ is cyclic, with d points (counting multiplicities) in its spectrum and it satisfies the quadrature identity:

$$\langle P(T_d)\xi, Q(T_d)\xi\rangle_{H_d} = \langle P(T)\xi, Q(T)\xi\rangle_H, \tag{14}$$

for any pair of polynomials $P, Q \in \mathbf{C}[z],\ \deg(P) \leq d,\ \deg(Q) \leq d-1$.

c). $\sigma(T_d) \subset \overline{W(T)}$.

We leave the standard proof to the reader.

A difficult and fundamental question of approximation theory is the location of $\sigma(T_d)$, or equivalently of the zero set of the "orthogonal" polynomial p_d. Along these lines, inclusion c) above is a generalization of Fejér's classical theorem which asserts that a complex orthogonal polynomial has its zeroes contained in the convex hull of the support of the underlying measure (in $\mathbf{C}$), see [12].

At this point, several convergence results are easily available. The notations and non-vanishing assumption are those of Proposition 2.1.

Theorem 2.5. *The sequence of rational functions $F_d(z, \overline{w})$ converges uniformly to $F(z, \overline{w})$, on compact subsets of $[\tilde{\mathbf{C}} \setminus \overline{W(T)}]^2$.*

Proof. Let $a \in \mathbf{C} \setminus \overline{W(T)}$ and let $L(z) = \Re(\alpha z + \beta)$ be a real linear functional which separates the compact convex set $\overline{W(T)}$ from a: $L(a) < 0 < L(z),\ z \in \overline{W(T)}$. Let $\epsilon > 0$ be small, so that $L(a + z) < 0$ for all $|z| < \epsilon$.

The operators $\alpha T + \beta$ and $\alpha T_d + \beta$ have their numerical range contained in the right half-plane $\mathbf{C}_+$, hence, in virtue of von Neumann's inequality (cf. [25] §154),

$$\|(\alpha S + \beta - (\alpha z + \beta))^{-1}\| \leq dist(\alpha z + \beta, \mathbf{C}_+)^{-1},$$

where $S = T$ or $S = T_d,\ d \geq 1$, and $|z - a| < \epsilon$.

There exists therefore a constant C with the property that:

$$\|(S - z)^{-1}\| \leq C; \quad S = T \text{ or } S = T_d,\ d \geq 1,\ |z - a| < \epsilon.$$

Then, whenever $|z - a| < \min(\epsilon, C)$, the familiar Neumann series expansion:

$$(S - z)^{-1} = (S - a - (z - a))^{-1} = \sum_{n=0}^{\infty}(z - a)^n (S - a)^{-n-1},$$

converges uniformly in the operator norm topology.

On the other hand,

$$(T_d - a)^{-1} - (T - a)^{-1} = (T_d - a)^{-1}(T - T_d)(T - a)^{-1}$$

converges strongly to zero, and hence $(T_d - a)^{-n}$ converges strongly to $(T - a)^{-n}$ for every $n \geq 1$.

In conclusion, $so-\lim_{d\to\infty}((T_d - z)^{-1} - (T - z)^{-1}) = 0$, uniformly in z belonging to an open ball centered at a. □

Often, the Padé approximation of a meromorphic function f holds beyond the radius of convergence, outside the poles of f, up to an inner essential barrier, see for instance [23] or [5]. The next corollary of Theorem 2.5 illustrates such a situation. Some functional examples will be considered in the next sections.

Corollary 2.6. *Let $K \in L(H)$ be a compact operator and let $W = \overline{W(T + K)}$. Then $\sigma(T)\setminus W$ is a discrete set in $\mathbf{C}\setminus W$ and the sequence F_d converges uniformly to F, on compact subsets of $[\tilde{\mathbf{C}} \setminus (W \cup \sigma(T))]^2$.*

Proof. The fact that $\sigma(T) \setminus W$ is a discrete set belongs to general perturbation theory, see [18].

Let $a \in \mathbf{C} \setminus (W \cup \sigma(T))$. Then the operator $T - a$ is invertible, and so are $T+K-a$, T_d+K_d-a; moreover, according to the preceding proof, $so-\lim_{d\to\infty}(T_d+K_d - a)^{-1} = (T + K - a)^{-1}$.

Since K is a compact operator we have $\lim_{d\to\infty} K_d = K$, and, $\lim_{d\to\infty}(T_d + K_d - a)^{-1}K = (T + K - a)^{-1}K$, where both limits are taken in the norm topology. Consequently, $\lim_{d\to\infty}(T_d + K_d - a)^{-1}K_d = (T + K - a)^{-1}K$. Similarly, because $so - \lim_{d\to\infty} T_d^* = T^*$, we obtain: $\lim_{d\to\infty} K_d(T_d + K_d - a)^{-1} = K(T + K - a)^{-1}$.

Next we write

$$\begin{gathered}(T - a)^{-1} = (T + K - a - K)^{-1} = \\ (T + K - a)^{-1}[I - K(T + K - a)^{-1}]^{-1} = \\ [I - (T + K - a)^{-1}K]^{-1}(T + K - a)^{-1},\end{gathered}$$

and we infer from this representation that there exists d_0 with the property that $(T_d - a)^{-1}$ is invertible for all $d \geq d_0$, with inverse:

$$\begin{gathered}(T_d - a)^{-1} = (T_d + K_d - a)^{-1}[I - K_d(T_d + K_d - a)^{-1}]^{-1} = \\ [I - (T_d + K_d - a)^{-1}K_d]^{-1}(T_d + K_d - a)^{-1}.\end{gathered}$$

The latter formulas also give $so - \lim_{d\to\infty}(T_d - z)^{-1} = (T - z)^{-1}$, uniformly in z, $|z - a| < \epsilon$, for a small, but positive ϵ. Then we continue as in the proof of the theorem. □

The particular case $T + K = 0$ in the preceding corollary yields the following result.

Corollary 2.7. *Assume that the operator T is compact. Then F_d converges to F uniformly on compact subsets of $[\tilde{\mathbf{C}} \setminus \sigma(T)]^2$.*

Corollaries 2.6 and 2.7 illustrate Montessus de Ballore convergence phenomena. For more details about them we refer to [5], [6].

Let us mention that the analytic function $F(z, \overline{w})$ detects the isolated points of finite multiplicity in the spectrum of T, such are for instance the discrete sets of points appearing in Corollaries 2.6 and 2.7. Indeed, let F be analytic in the punctured bidisk $[D(a, \epsilon) \setminus \{a\}]^2$, and assume that there exists a positive integer n so that, for all $w \in D(a, \epsilon) \setminus \{a\}$,

$$\lim_{z\to a}(z - a)^{n+1}F(z, \overline{w}) = 0.$$

Then the spectral space of T, corresponding to the isolated component $\{a\} \in \sigma(T)$, is finite dimensional, of dimension not exceeding $n + 1$. We leave the details to the reader.

In complete analogy with the classical theory of orthogonal polynomials we derive below a formula for the remainder in our approximation problem. We will deduce then from it a weaker, but more general, convergence in measure statement. The notation is unchanged.

First, remark that, for large values of $|z|$, we have:

$$p_d(z)(T - z)^{-1}\xi = [p_d(z) - p_d(T)](T - z)^{-1}\xi + p_d(T)(T - z)^{-1}\xi =$$

$$[p_d(z) - p_d(T_d)](T_d - z)^{-1}\xi + p_d(T)(T - z)^{-1}\xi =$$

$$p_d(z)(T_d - z)^{-1}\xi + p_d(T)(T - z)^{-1}\xi.$$

In adition, we note the identity: $p_d(T)\xi = (1 - \pi_d)T^d\xi$. All these computations can be restated in the following lemma.

Lemma 2.8. *Let $z \notin \sigma(T) \cup \sigma(T_d)$. Then:*

$$(T - z)^{-1}\xi - (T_d - z)^{-1}\xi = (T - z)^{-1}\frac{(1 - \pi_d)T^d\xi}{p_d(z)}. \tag{15}$$

As expected, the asymptotics of the sequence $\|(1 - \pi_d)T^d\xi\|$, $d \geq 1$, decides the rate of convergence of our approximation process. A general, and not very illuminating, way of computing $\|(1 - \pi_d)T^d\xi\|$ is to consider the matrix associated to T, with respect to the orthonormal basis $\frac{\xi}{\|\xi\|}, \frac{p_1(T)\xi}{\|p_1(T)\xi\|}, \ldots$:

$$T \sim \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ 0 & a_{32} & a_{33} & \cdots \\ 0 & 0 & a_{43} & \cdots \\ \vdots & & \vdots & \end{pmatrix}.$$

Condition (6) is then equivalent to the fact that $a_{p+1,p} \neq 0$ for all $p \geq 1$. By keeping track of the first column in the matrix associated to T^d we immediately obtain:

$$\|(1-\pi_d)T^d\xi\| = |a_{21}a_{32}\dots a_{d+1,d}|. \tag{16}$$

Recall that very similar formulas are known in the theory of orthogonal polynomials, see [29].

Next we present a typical application of a lemma due to H. Cartan to the convergence in measure of the sequence F_d, which in general is valid beyond the range of applicability of Theorem 2.5. For full details about this method of proof and its numerous ramifications, as for instance convergence in capacity rather than planar measure, we refer to [2] Chapter 6 or to the ferences in [6].

Theorem 2.9. *Let T be an operator with cyclic vector ξ satisfying*

$$\lim_{d\to\infty} \|(1-\pi_d)T^d\xi\|^{1/d} = 0$$

and let ϵ, δ be positive numbers.

There exists d_0 depending on ϵ, δ with the property that, for every $d \geq d_0$, there exists a measurable set $E_d \subset \mathbf{C}$ of area $|E_d| \leq \pi\delta^2$ and such that:

$$\|(T-z)^{-1}\xi - (T_d - z)^{-1}\xi\| \leq \|(T-z)^{-1}\|\epsilon^d, \tag{17}$$

for all $z \notin \sigma(T) \cup E_d$.

Proof. According to Cartan's lemma, for each monic polynomial of degree d, in particular for p_d, there exists a measurable set E_d of area $|E_d| \leq \pi\delta^2$, such that:

$$|p_d(z)| > \delta^d, \quad z \notin E_d.$$

It remains to choose d_0 with the property that $d \geq d_0$ implies $\|(1-\pi_d)T^d\xi\|^{1/d} \leq \epsilon\delta$. Then Lemma 2.8 yields the desired estimate. □

A polarization of estimate (17) will give the convergence in measure $F_d(z, \overline{w}) \to F(z, \overline{w})$, for $z, w \notin \sigma(T)$.

To construct examples of operators as in the theorem above, pick a subdiagonal sequence in the preceding matricial decomposition of T so that

$$\lim_{d\to\infty} |a_{21}a_{32}\dots a_{d+1,d}|^{1/d} \to 0.$$

For the rest choose independently the upper triangular elements of T, including by convention among them the diagonal. Since we can regard T as a perturbation of an upper triangular matrix D by a compact subdiagonal, the essential spectrum of T will be equal to the essential spectrum of D, see [18], and hence it can be arbitrary.

3. Subnormal operators

If T is a subnormal operator, then the abstract Hilbert space objects described in the preceding section have a function theoretic counterpart which is a slight generalization of the classical Padé approximation framework. The present section contains a couple of examples of this kind.

1. Let $T = T^*$ be a *selfadjoint operator* with cyclic vector ξ. Then the spectral theorem gives the realization $T = M_x$ of T as the multiplication operator with the variable $x \in \mathbf{R}$, on the Hilbert space $H = L^2(\mu)$, where μ is a positive, compactly supported Borel measure on $\mathbf{R}$. In this representation $\xi = \mathbf{1}$, the function identically equal to 1. Therefore, the function (1) to be approximated from its germ at infinity is:

$$F(z,\overline{w}) = \int_{\mathbf{R}} \frac{d\mu(t)}{(t-z)(t-\overline{w})}, \quad z,w \notin \mathbf{R}. \tag{18}$$

Assume that the closed support of μ is contained in a compact interval: $supp(\mu) \subset [a,b]$. Then $\sigma(T) = supp(\mu) \subset [a,b]$ and moreover, the very definition of the numerical range yields: $\overline{W(T)} \subset [a,b]$. Note that in this situation,

$$b_{mn} = \langle T^m\xi, T^n\xi\rangle = \langle T^{m+n}\xi,\xi\rangle,$$

hence $b_{mn} = b_{m'n'}$, whenever $m+n = m'+n'$. Therefore p_d is the monic orthogonal polynomial of degree d with respect to the measure μ, and our approximant, when evaluated at infinity in one variable, coincides with the classical $[n-1,n]$ Padé approximant:

$$\lim_{w\to\infty} \overline{w}F_d(z,\overline{w}) = -\langle (T_d - z)^{-1}\xi,\xi\rangle,$$

and

$$\langle (T-z)^{-1}\xi,\xi\rangle - \langle (T_d-z)^{-1}\xi,\xi\rangle = \sum_{k=2d}^{\infty} \frac{\gamma_k}{z^{k+1}}.$$

Thus Theorem 2.5 generalizes Markov's theorem, see [23], §68.

Next we consider a simple example which illustrates Corollary 2.6. Besides the measure μ supported by the real line, let us consider a finite atomic, positive measure ν, supported by $\mathbf{C}\setminus\mathbf{R}$. Let us denote $supp(\nu) = \{a_1, a_2, \dots, a_n\}$ and $\nu_i = \nu(\{a_i\})$. The Hilbert space $H = L^2(\mu+\nu) = L^2(\mu)\oplus L^2(\nu)$ still carries $T = M_z$ as a linear bounded operator with cyclic vector $\xi = \mathbf{1}$. Accordingly,

$$F(z,\overline{w}) = \int_{\mathbf{R}} \frac{d\mu(t)}{(t-z)(t-\overline{w})} + \sum_{i=1}^{n} \frac{\nu_i}{(a_i - z)(\overline{a_i} - \overline{w})}.$$

By considering the operator M_z on H as a direct sum of M_x on $L^2(\mu)$ and the finite rank perturbation M_z on $L^2(\nu)$, Corollary 2.6 states that F_d converges uniformly to F on compact subsets of the complex plane which avoid the set $[a,b] \cup \{a_1, a_2, \dots, a_n\}$. Again, letting $w = \infty$ and normalizing the functions by the factor $\overline{w}$, we obtain that the sequence of rational functions $\langle (T_d - z)^{-1}\xi,\xi\rangle$ converges in the same domain to the function:

$$\int_{\mathbf{R}} \frac{d\mu(t)}{t-z} + \sum_{i=1}^{n} \frac{\nu_i}{a_i - z}.$$

Actually in the above example we can take $n = \infty$, with the only necessary additional assumptions:

$$\lim_{i\to\infty} a_i = 0, \qquad \sum_{i=1}^{\infty} \nu_i < \infty.$$

For classical approximation results of this type we refer to [2], Part I and the references cited there.

2. The case when T is *subnormal* is very similar. Let μ be a positive Borel measure compactly supported by the complex plane and let $H = P^2(\mu)$ be the closure in $L^2(\mu)$ of the space $\mathbf{C}[z]$ of complex polynomials. The multiplication operator $T = M_z$ is then the typical cyclic subnormal operator, with cyclic vector $\xi = \mathbf{1}$, see [4]. The function F is this time:

$$F(z,\overline{w}) = \int_{\mathbf{C}} \frac{d\mu(\zeta)}{(\zeta - z)(\overline{\zeta} - \overline{w})},$$

but $\overline{w}F_d(z,\overline{w})|_{w=\infty}$ may fail to be a $[d-1,d]$ approximant of $\overline{w}F(z,\overline{w})|_{w=\infty}$. In any case,

$$\overline{w}F_d(z,\overline{w})|_{w=\infty} - \overline{w}F(z,\overline{w})|_{w=\infty} = \sum_{k=d+1}^{\infty} \frac{\gamma_k}{z^{k+1}}.$$

The closed numerical range $\overline{W(M_z)}$ can in this case be identified with the closed convex hull of $supp(\mu)$:

$$W = \overline{W(M_z)} = \overline{co\ supp(\mu)}.$$

The convergence in Theorem 2.5 and its corollary holds then ouside W, respectively W union a discrete set.

We leave the details to the interested reader.

4. Extremal hyponormal operators

In this section we return to the original approximation problem of [15]. Let us remind first the specific form of the function F in this situation. The terminology will be borrowed freely from [15] and [24], except a switch from an operator to its adjoint, respectively from z to $\overline{z}$, which we hope will cause no serious confusion to the reader.

Let Ω be a bounded planar domain, and let $T_\Omega \in L(M)$ be the unique hyponormal operator of rank-one self-commutator $[T_\Omega{}^*, T_\Omega] = \xi \otimes \xi$, having its principal function equal to the characteristic function of Ω. Then $\sigma(T_\Omega) = \overline{\Omega}$ and the essential spectrum of T_Ω is the boundary of the domain, see [20]. Let, as before $H = \bigvee_{n=0}^{\infty} T_\Omega{}^{*n}\xi$ and recall that H can be a proper closed subspace of M, see [24]. To be consistent with our previous paragraphs we denote $T = T_\Omega{}^*|H$, and we regard T as a linear continuous operator on H, with cyclic vector ξ.

Then there exists a positive definite kernel $H(z,\overline{w})$, analytic and integrable in $z \in \Omega$, anti-analytic and integrable in $w \in \Omega$ and such that our series F has the representations, for large values of $|z|, |w|$:

$$F(z,\overline{w}) = \sum_{m,n=0}^{\infty} \frac{b_{mn}}{z^{m+1}\overline{w}^{n+1}} = \langle (T-z)^{-1}\xi, (T-w)^{-1}\xi\rangle =$$

$$1 - exp[\frac{-1}{\pi}\int_\Omega \frac{dA(\zeta)}{(\zeta - z)(\overline{\zeta} - \overline{w})}] =$$

$$\int_\Omega\int_\Omega \frac{H(u,\overline{v})dA(u)dA(v)}{(u-z)(\overline{v}-\overline{w})},$$

see [24], [16]. All integrals above are taken with respect to the area measure dA.

Further on, assume that Ω is a *generalized quadrature domain*, in the sense that there exists a signed measure μ, supported by a compact subset ω of Ω (or even $\overline{\Omega}$) with the property:

$$\int_\Omega f dA = \int_\omega f d\mu, \quad f \in AL^1(\Omega), \tag{19}$$

where $AL^1(\Omega)$ is the space of all integrable, analytic functions in Ω. The domain Ω is called a *quadrature domain* if a representation formula (19) exists with ω reduced to a finite set and μ possibly replaced by a distribution. It is known that quadrature domains are very rigid, for instance their boundary is given by an irreducible polynomial equation. For details see [28].

The main result of [24] asserts that Ω is a quadrature domain if and only if $dim(H) < \infty$; that means, in our previous notation, that there exists $d \geq 1$ with the property $F_{d+k} = F$ for all $k \geq 0$. Thus, *quadrature domains correspond*, in the above framework, *exactly to the case when the Padé approximation scheme $F_d \to F$ becomes stationary for large d.* This fact was further exploited in [13] and [15].

Returning to a generalized quadrature domain Ω, we remark that the representations of F can be analytically continued in the explicit form:

$$\tilde{F}(z,\overline{w}) = \int_\omega \int_\omega \frac{H(u,\overline{v})d\mu(u)d\mu(v)}{(u-z)(\overline{v}-\overline{w})}, \quad z,w \notin \omega. \tag{20}$$

By passing to the single variable, Hilbert space valued picture we can factor the kernel H as $H(u,\overline{v}) = \langle h(u), h(v)\rangle$, where $h : \Omega \longrightarrow H$ is an integrable, analytic function. By reading formula (20) on its first factor we find:

$$(T-z)^{-1}\xi = \int_\omega \frac{h(u)d\mu(u)}{u-z}, \quad z \notin \sigma(T). \tag{21}$$

We recall that in this case the boundary of Ω is real analytic, see [16].

The only interesting case is when $\omega \neq \overline{\Omega}$; then the mere existence and the properties of the analytic continuation of the localized resolvent $(T-z)^{-1}\xi$ have far reaching consequences. We state below one of them. For a comprehensive account of the notion of capacity we refer for instance to [14] or [26].

Lemma 4.1. *Assume that Ω is a generalized quadrature domain (19) which is not a quadrature domain. Then the "orthogonal" polynomials p_d satisfy:*

$$\limsup \|p_d(T)\xi\|^{1/d} \leq c, \tag{22}$$

where $c = cap(\omega)$ is the capacity of the support of the quadrature measure μ.

Proof. The fact that Ω is not a quadrature domain is equivalent to condition (6), see [24].

It is known, [14], that there exists a sequence of monic polynomials r_d,
$\deg(r_d) = d$, with the property $c = \lim_{d\to\infty} \|r_d\|_{\infty,\omega}^{1/d}$.

By its very definition, the polynomial p_d satisfies the minimality condition
$\|p_d(T)\xi\| \leq \|r_d(T)\xi\|$. On the other hand, the explicit analytic extension formula (21) gives, via the Riesz-Dunford functional calculus:

$$\|r_d(T)\xi\| = \|2\pi \int_\omega r_d(u)h(u)d\mu(u)\| \leq C\|r_d\|_{\infty,\omega},$$

where $C = 2\pi\|h\|_{\infty,\omega}\|\mu\|$. □

Returning now to the remainder estimates of Theorem 2.9 we can state the following approximation result.

Theorem 4.2. *Let Ω be a generalized quadrature domain, whose quadrature measure is supported by a compact set ω of capacity zero.*

Let ϵ, δ be fixed positive numbers and let K be a compact subset of $\mathbf{C} \setminus \omega$. Then there exists $d_0 \in \mathbf{N}$, depending on all these parameters, so that for every $d \geq d_0$ there exists a measurable set E_d of area $|E_d| \leq \pi\delta^2$, with the property:

$$\|(T-z)^{-1}\xi - (T_d - z)^{-1}\xi\| \leq \epsilon, \quad z \in K \setminus E_d.$$

By an abuse of notation we have denoted above by $(T-z)^{-1}\xi$ the analytic continuation (21) of that function on points $z \in \overline{\Omega} \setminus \omega$.

Proof. If Ω is a quadrature domain, then there exists d_0 such that $(T-z)^{-1}\xi = (T_d - z)^{-1}\xi$ for $d \geq d_0$.

Let Ω be a domain as in the statement which is not a quadrature domain. We can assume after a rescaling that $\Omega \subset D(0, 1/2)$. Then the spectrum of the operator T is contained in the closed disk $\overline{D(0,1/2)}$, and being a hyponormal operator, its spectral radius coincides with its norm, so $\|T\| \leq 1/2$. On the other hand the polynomial p_d is monic and has roots $a_i, 1 \leq i \leq d$, inside the same disk, hence

$$\|p_d(T)\| \leq \prod_{i=1}^{d}(\|T\| + \|a_i\|) \leq 1.$$

Fix a point $z_i \in K$. Since the underlying Hilbert space is spanned by $T^n\xi$, with $n \geq 0$, there exists an open neighbourhood U_i of z_i and a polynomial $r_i(z)$ with the property:

$$\|(T-z)^{-1}\xi - r_i(T)\xi\| \leq \frac{\epsilon}{2}, \quad z \in U_i.$$

Choose a finite open covering $K \subset \cup_{i=1}^m U_i$.

Accordingly, we can estimate the numerator in the remainder formula (15) as follows:

$$\|p_d(T)(T-z)^{-1}\xi\| \leq \frac{\epsilon}{2} + \|r_i(T)\|\|p_d(T)\xi\|, \quad z \in U_i.$$

Let $\rho = \max_{i=1}^m \|r_i(T)\|$.

In virtue of the preceding lemma there exists d_0 so that the second term is majorized by:

$$\|p_d(T)\xi\| \leq \frac{\epsilon\delta^d}{2\rho}, \quad d \geq d_0.$$

And from this point on we can repeat the proof of Theorem 2.9 .

□

Outside the set Ω, Theorem 4.2 is actually covered by Theorem 2.9. The point is that the convergence in Theorem 4.2 holds, in measure, on $\Omega \setminus \omega$. Examples of generalized quadrature domains with a positive representing measure supported by a set of capacity zero can easily be constructed by an outward balayage process, see [28] and the references cited there.

Example 4.3. *(A smooth, real analytic, convex domain).*

As concerns the practical matters of [15], let us consider the ideal situation of a convex bounded domain Ω with smooth real analytic boundary. With the above notation, there exists a relatively compact subdomain $\omega \subset \Omega$ such that:

$$\Omega \setminus \overline{\omega} = \{z \in \mathbf{C} \setminus \overline{\omega};\ \|(T-z)^{-1}\xi\| > 1\},$$

see [15]. Since, according to Theorem 2.5, the sequence of rational approximants $(T_d - z)^{-1}\xi$ converges uniformly to $(T-z)^{-1}\xi$ for points $z \notin \overline{\Omega}$, it is legitimate to approximate the boundary Γ of Ω by the sets

$$\Gamma_d(\epsilon) = \{z \in \mathbf{C} \setminus \overline{\omega};\ \|(T_d - z)^{-1}\xi\| = 1 - \epsilon\}, \quad \epsilon > 0.$$

Indeed, for a fixed $\epsilon > 0$, when d tends to ∞, $\Gamma_d(\epsilon)$ converges in the Hausdorff topology to

$$\Gamma(\epsilon) = \{z \in \mathbf{C} \setminus \overline{\omega};\ \|(T - z)^{-1}\xi\| = 1 - \epsilon\}, \quad 0 < \epsilon < 1,$$

and the latter converges to Γ as ϵ decreases to 0, due to the local uniform continuity of the function $\|(T_d - z)^{-1}\xi\|,\ z \notin \overline{\omega}$.

The numerical experiments recorded in [15] validate the above explanation.

We close this section by a simple example, showing that the analytic continuation configuration of the series $F(z, \overline{w})$ can be quite independent of the boundary $|z| = |w| = R = \|T\|$ arising from condition (2).

Example 4.4. *(The ellipse).*

Let $0 < r < 1$ be a fixed real number and let $U \in L(H)$ be the unilateral shift $U(e_n) = e_{n+1}$, where $e_0, e_1, \dots$, is an orthonormal basis of the Hilbert space H. Then $[U^*, U] = e_0 \otimes e_0$, so that $U = T_{\mathbf{D}}$ is the hyponormal operator associated as before to the unit disk $\mathbf{D}$.

We consider the linear combination $U + rU^*$, so that:

$$[U^* + rU, U + rU^*] = (1 - r^2) e_0 \otimes e_0.$$

A simple inductive argument shows that e_0 is a cyclic vector for $U + rU^*$, as well as for its adjoint. On the other hand, the functoriality property of the principal function, see [20] §X.3.11, shows that this operator is associated to an ellipse, in the following precise sense: $U + rU^* = T_E$, where:

$$E = \{\zeta \in \mathbf{C};\ \zeta = z + r\overline{z},\ |z| < 1\} = \{x + iy;\ (\frac{x}{1+r})^2 + (\frac{y}{1-r})^2 < 1\}.$$

Thus, in our notation, we consider the operator $T = T_E^* = U^* + rU$, with cyclic vector e_0 and corresponding power series:

$$\langle (T - z)^{-1} e_0, (T - w)^{-1} e_0 \rangle = F(z, \overline{w}).$$

As a co-hyponormal operator with spectrum equal to the ellipse E, T has spectral radius equal to its norm: $R = \|T\| = 1 + r$. Thus, *a priori*, the preceding series is convergent for $\min\{|z|, |w|\} > 1 + r$.

In reality, see [16], the series converges for $z, w \notin [-2\sqrt{r}, 2\sqrt{r}]$, the straight line segment joining the foci of the ellipse. Actually , even an explicit analytic continuation of this double series is given in [16].

As a second byproduct of these computations we can explicitly find the polynomials p_d in the approximation process. Indeed, the space H_d is generated by the vectors $T^k e_0,\ k \le d$, hence, by induction $H_d = \bigvee_{k=0}^{d-1} e_k$. Thus the matrix T_d is:

$$T_d = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ r & 0 & 1 & \dots & 0 \\ 0 & r & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & r & 0 \end{pmatrix}.$$

Up to a normalization we can then identify, via the three term recurrence relation, the characteristic polynomial $p_d(z) = \det(z - T_d)$ with a Chebyshev polynomial of the second kind:

$$p_d(z) = r^{d/2} U_d\left(\frac{z}{2\sqrt{r}}\right), \quad d \ge 1.$$

Indeed, $p_{d+1}(z) = zp_d(z) - rp_{d-1}(z)$, with the initial data $p_1(z) = z$, $p_2(z) = z^2 - r$, while $U_{d+1}(z) = 2U_d(z) - U_{d-1}(z)$, and $U_1(z) = 2z,\ U_2(z) = 4z^2 - 1$, see [10].

In particular this shows that the zeroes of the polynomials p_d are located on the straight line segment joining the two foci of the ellipse E. In view of represeantation (21) and the known estimate $\int_{-1}^{1} |U_d(x)| dx \le 2$ one can prove as in [15] that the analytic extensions of the functions $F_d(z, \overline{w})$ converge locally and uniformly to the analytic extension of $F(z, \overline{w})$, as soon as

$$z, w \in \{\zeta \in \mathbf{C};\ dist(\zeta, [-2\sqrt{r}, 2\sqrt{r}]) > \sqrt{r}\}.$$

We do not expand here this argument, but refer to [15] for details.

5. Stability questions

In this section we discuss the continuous dependence of the pair of polynomials (p_d, q_{d-1}) as functions of the Taylor coefficients b_{mn} of the original series (1). For a fixed degree d, and under the non-degeneration assumption (6), simple linear algebra arguments give an explicit, real analytic dependence of (p_d, q_{d-1}) on the entries $b_{mn},\ m, n \le d$. We analyze then a case when the dependence is uniformly continuous with respect to d.

First some additional notation. Fix a degree $d \ge 1$, and normalize the series $F(z, \overline{z})$ so that the constant R in (2) is equal to one, or equivalently the associated operator T is a contraction: $\|T\| \le 1$. The standard orthonormal basis of $l^2(\mathbf{N})$ will be denoted by $e_i,\ i \ge 1$. We regard then the matrix (b_{mn}) as acting on $l^2(\mathbf{N})$; it will be convenient to denote:

$$B_d = (b_{mn})_{m,n=0}^{d},$$

and regard it as an operator acting on the space spanned by $e_0, e_1, \dots, e_d$. We assume that condition (6) holds, that is $\det B_d \ne 0$. Therefore B_d is an invertible, positive $(d+1) \times (d+1)$ matrix. In general, for a matrix A, we denote by A_{ij} its entries.

With these preparations we can state the explicit form of the coefficients of the "orthogonal" polynomial $p_d(z) = z^d + c_{d-1} z^{d-1} + \dots + c_0$.

Proposition 5.1. *Assume that* $\det B_d \ne 0$. *Then:*

$$c_k = \frac{(B_d^{-1})_{kd}}{(B_d^{-1})_{dd}}, \qquad 0 \le k \le d. \tag{23}$$

Proof. We will work in the Hilbert subspace K of $l^2(\mathbf{N})$, generated by $e_0, e_1, \dots, e_d$. According to Corollary 2.2, we have to find the vector $c \in K, c_d = \langle c, e_d \rangle = 1$, which minimizes the quotient:

$$\frac{\langle B_d x, x \rangle}{|x_d|^2} \geq \langle B_d c, c \rangle = L, \quad x_d \neq 0.$$

This in turn is equivalent to maximizing the quantity:

$$\frac{\langle (e_d \otimes e_d) x, x \rangle}{\langle B_d x, x \rangle} \leq L^{-1}.$$

By denoting $y = \sqrt{B_d} x$, we have to find the unit vector $\sqrt{B_d} c / \| \sqrt{B_d} c \|$ which maximizes

$$\langle \sqrt{B_d^{-1}} (e_d \otimes e_d) \sqrt{B_d^{-1}} y, y \rangle = \langle (\sqrt{B_d^{-1}} e_d \otimes \sqrt{B_d^{-1}} e_d) y, y \rangle.$$

But this vector is necessarily a scalar multiple of $\sqrt{B_d^{-1}} e_d$. Therefore, c is a multiple of $B_d^{-1} e_d$. The assumption $c_d = 1$ yields then the result. $\square$

The above explicit formulas for the coefficients c_k have the following immediate consequence.

Corollary 5.2. *Assume that $0 < \gamma \leq B_d$, $d \geq 1$, for a constant γ. Then for each $\epsilon > 0$ there exists $\delta > 0$, so that for any $d \geq 1$ and any other matrix $\tilde{B}$ satisfying relation (2) (with $R = 1$) and $\| B_d - \tilde{B}_d \| \leq \delta$, the coefficients of the associated polynomials $p_d, \tilde{p}_d$ satisfy:*

$$\sum_{k=0}^{d-1} |c_k - \tilde{c}_k|^2 \leq \epsilon^2.$$

Unfortunately, Corollary 5.2 is not applicable to the examples considered in Section 4, because, according to formula (20):

$$b_{mn} = \int_\Omega \int_\Omega u^m \overline{v}^n H(u, \overline{v}) dA(u) dA(v).$$

On the other hand, our normalization assumption $\|T\| \leq 1$ implies $\Omega \subset D(0,1)$, so by Lebesgue dominated convergence theorem we find
$\lim_{n \to \infty} b_{nn} = 0$, which contradicts the lower bound assumption in the statement of Corollary 5.2.

However, there are many simple examples of operators T which fulfil the lower bound condition in Corollary 5.2. For instance, any *weighted shift*

$$T(e_k) = \lambda_k e_{k+1}, \quad k \geq 0,$$

whose weights satisfy $|\lambda_k| \leq 1$, $k \geq 1$, and $\prod_{k=1}^{\infty} |\lambda_k| \geq c > 0$, is such an example.

Indeed, in this case $b_{mn} = 0$ for $m \neq n$, and $b_{nn} = \lambda_0 \lambda_1 \dots \lambda_n$. Incidentally, we remark that this is a one-dimensional situation, in the radial direction:

$$F(z, \overline{z}) = \sum_{n=0}^{\infty} \frac{b_{nn}}{|z|^{2n}}.$$

A specialization of the main results of Section 2 for this class of series is then within reach in purely function theoretic terms. For instance, one easily cheks that the approximants of $F(z,\overline{z})$ are exactly the partial Taylor series at infinity:

$$F_d(z,\overline{z}) = \sum_{n=0}^{d} \frac{b_{nn}}{|z|^{2n}},$$

with $p_d(z) = z^d,\ d \geq 1$.

For such series, the uniformity in d asserted by Corollary 5.2 becomes easy to verify.

Speaking about weighted shifts, we end this section with a few remarks arising from studying such an operator.

Example 5.3. *(The annulus).*

Let $0 < r < 1$ be a fixed real number, and let $A = \{z \in \mathbf{C};\ r < |z| < 1\}$ be the annulus of radii $r, 1$. In the light of Section 4 above, the corresponding hyponormal operator $T_A \in L(M)$, of rank-one self-commutator $[T_A^*, T_A] = \xi \otimes \xi$ and principal function equal to the characteristic function of A has spectrum equal to $\overline{A}$. In particular $\|T_A\| = 1$. We set as before $T = T_A^*|H$, where $H = \bigvee_{n=0}^{\infty} T_A^{*n}\xi$.

The associated power series (3) is in this case:

$$F(z,\overline{z}) = 1 - \frac{1-|z|^{-2}}{1-r^2|z|^{-2}} = \frac{1-r^2}{|z|^2-r^2} =$$

$$\frac{1-r^2}{|z|^2} + \frac{(1-r^2)r^2}{|z|^4} + \frac{(1-r^2)r^4}{|z|^6} + \ldots, \quad |z| > 1,$$

see [16].

But this immediately yields $\langle T^n\xi, T^m\xi\rangle = 0$ if $n \neq m$, and $\|T^n\xi\|^2 = (1 - r^2)r^{2n},\ n \geq 0$. Consequently we are led to the identification $\xi = \sqrt{1-r^2}e_0$, and $Te_n = re_{n+1}, n \geq 0$, where $e_n,\ n \geq 0$, is an orthonormal basis of the Hilbert space H.

Thus $T = rU$ is a multiple of the unilateral shift U. In particular, $\|T\| = r < 1 = \|T_A\|$. This shows that the space $H = \bigvee_{n=0}^{\infty} T_A^{*n}\xi$ is properly contained in the original Hilbert space M where T_A acts. Second, we remark that the spectrum of T (i.e. the disk $r\overline{\mathbf{D}}$) is different from the spectrum of T_A (the closed annulus $\overline{A}$).

In conclusion, if we want to reconstruct the annulus A with the aid of its moments, as explained in [15], the sequence of approximants $F_d(z,\overline{z}) - 1$ will stably converge to the defining equation $F(z,\overline{z}) - 1$ of the outer boundary, on compact subsets of $\mathbf{C} \setminus r\overline{\mathbf{D}}$, but not beyond the circle $|z| = r$.

To make such an approximation work for the inner boundary $|z| = r$ we must first change the coordinate from z to $1/z$, then analytically extend the germ at infinity of the local resolvent of $T_{A^{-1}}^*$, and then repeat the same process.

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