A SHORT INTRODUCTION TO THE IDEA OF PROOF

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1. Basic information

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Exercise 1.1. Take an 8x8 chessboard and remove one square from the upper-right-hand corner. Can you cover it with 2x1 dominoes so that no two dominoes overlap?

2. What is a proof?

Loosely defined, a proof is a rigorous mathematical argument that demonstrates that some proposition is true. They are how (pure) mathematics proceeds; they are the language mathematicians use to convince each other of ideas. Specifically, a proof of a given statement is a series of axioms or previously established results, linked together by logical deductions, which at its end demonstrates the validity of the given statement.

Proving things in a rigorous fashion is a difficult trick to get the hang of; it will likely take you a while, and your homeworks will be (perhaps) full of red ink. Do not fret! Everyone is in the same boat as you, and it is a hard thing to initially learn to do. The best way to get a feel for how proofs work is to look at examples (say online at mathworld/in Apostol/in lecture/in recitation) and through trial and experimentation. If you are unsure if something is a valid proof or method of proof, ask me and I'll be glad to look it over.

Exercise 2.1. Again, take an 8x8 chessboard and (this time) remove one square from the upper-right-hand corner and one from the bottom-left-hand corner. Can you cover this board with 2x1 dominoes so that no two dominoes overlap?

3. Common Mistakes

(Bad) things people often do when starting theoretical mathematics:

- **Not using words.** Mathematics, contrary to popular opinion, does not consist of long strings of formulae and numbers; it is made of words, sentences and paragraphs. State what you are doing, and write in words the logic you employ to move your proof along. Not that you have to explain everything: i.e. the move from $15x = 5y$ to $3x = y$ doesn't need a treatise. But if you had to put thought into a given step, it is a good bet that your readers will want you to explain what you did.

- **Using empirical evidence.** Testing things with examples is a good way to build intuition. But it usually does not prove a statement. (A counterexample would be when you're disproving something; there, examples suffice.) If you want to show that (say) the product of two odd numbers is odd, saying $3 \cdot 3 = 9; 7 \cdot 5 = 35$; QED is, um, not a proof. (You wouldn't believe how many times I've seen proofs like that.)
- **Messing up the contrapositive.** Suppose you want to show that, say, "all ravens are black." A logically equivalent statement you could show would be that "all non-black things are not ravens;" showing one of these statements is the same as showing the other. Abstractly: showing X implies Y for two statements X and Y is the same as showing that not-Y implies not-X. **DO NOT** make the mistake of thinking that $X \Rightarrow Y$ is the same as $\neg X \Rightarrow \neg Y$. This happens all the time, with frequently hilarious results. (note: hilarious results are usually things you want to avoid in math classes.)

Exercise 3.1. Now take an 8x8 chessboard and randomly remove a white square and a black square from somewhere on the board. Again, can you cover this board with 2x1 dominoes so that no two dominoes overlap?

4. Example Proofs; Methods of Proof

There are many methods of proof; here are a few examples.

Theorem 4.1. *If n is an integer and n^2 is even, then n^2 is a multiple of 4.*

Proof. **Direct Proof:**If n is an integer, it is either odd or even. If n is odd, we know that n^2 is also odd (as it is the product of two odd numbers.) So n must be even; thus, we can write $n = 2 \cdot k$ for some other integer k. Then $n^2 = (2 \cdot k)^2 = 4k^2$ which is a multiple of 4. $\square$

Direct proofs are (usually) boring.

Theorem 4.2.

$$1 + 2 + 3 + \ldots + n = \frac{(n)(n+1)}{2}$$

Proof. **Inductive Proof:** We note first that the above equality is trivially true when $n = 1$, as $1 = 2/2$. Now that we have established the base case, we proceed to the inductive step: assuming that the above equality holds for n, we attempt to prove that it must consequently hold for $n + 1$.

So: if the equality

$$1 + 2 + 3 + \ldots + n = \frac{(n)(n+1)}{2}$$

, holds, we have (adding $n + 1$ to both sides) the equality

$$1 + 2 + 3 + \ldots + n + n + 1 = \frac{(n)(n+1)}{2} + n + 1,$$

which is just

$$1 + 2 + 3 + \ldots n + 1 = \frac{(n)(n+1) + (2)(n+1)}{2} = \frac{(n+2)(n+1)}{2},$$

which is what we wanted to prove. $\square$

Inductive proofs are far cooler! (yay!) However, cooler things are usually dangerous. Spot the flaw in the following proof, due to G. Pólya:

Theorem 4.3. *All horses are the same color.*

Proof. We will prove that any finite collection of horses are all of the same color; as there are only finitely many horses in existence, this will suffice to prove our claim. We proceed by induction: first, note that any group consisting of just one horse is trivially a group in which all the horses are of the same color. This takes care of the base case.

So we are left with the inductive step: assume that all groups of horses of size n are all of the same color. We then want to show that any group of horses $\{1, 2, \ldots n+1\}$ must also all be of the same color. So: look at the group of horses $\{1, 2, \ldots n\}$: this is a group of size n, and thus a group all of the same color. Then look at the group $\{2, 3, \ldots n+1\}$; this is also a group of size n and thus also of the same color. As these two groups have highly nontrivial overlap, we can thus conclude that the coloring of the group $\{1, 2, \ldots n\}$ is the same as the group $\{2, 3, \ldots n+1\}$; so the entire group of horses is of the same color. So all horses are the same color. $\square$

Exercise 4.4. Suppose that you have a 10x10 chessboard (manufacturing defect, perhaps?) and suppose further that you also have a pile of 4x1 dominoes. Can you cover this board with 4x1 dominoes so that no two dominoes overlap? (Hint: look at your solutions to the previous exercises, and generalize.)