RECITATION 4: HIGHER ORDER DERIVATIVES; INTEGRALS (KINDA)

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1. Last Week's Homework

Last week's HW was fairly solid overall, with an average around 85; there were just a pair of things that cost many people some points.

(1) If you're using theorems, state them in your proof, and show that the conditions necessary to use that proof hold.. This lost a lot of people points on the first few questions. This may seem somewhat pedantic, but it's good form, and really improves the clarity of your work.

(2) For the problem where you had to show that (for f a polynomial)

$$f \text{ has } r \text{ roots} \Rightarrow f' \text{ has at least } r-1 \text{ roots},$$

many people made the mistake of assuming that the roots of f were all distinct! This is very often not the case (consider $f(x) = (x-13)^{10^100}$, for example).

2. Test Results

The test average for our section was around 67, std. dev. 15, which was comparable to the other sections; there were things that could have gone better, but no one did poorly enough to place themselves in jeopardy of failing the course (assuming, of course, that there are no further drops.) If you have questions about the scoring, or about the questions in general, feel free to come and talk to me after class.

3. New Theorems and Definitions

Just a few this week:

Theorem 3.1. *The (first) Fundamental Theorem of Calculus: suppose that f is integrable on $[a, x]$, for all $x \leq b$. Then choose any c in $[a, b]$, and define*

$$A(x) = \int_c^x f(t)dt$$

(where A is defined on the interval $[a, b]$).

Then A is differentiable on every $x \in (a, b)$ where f is continuous, and $A'(x) = f(x)$.

Theorem 3.2. *The (second) Fundamental Theorem of Calculus: suppose that f is continuous on (a, b), and pick any primitive P of f. Then for every $c, x \in (a, b)$,*

$$P(x) = P(c) + \int_c^x f(t)dt.$$

4. Worked problems

We study here an example of using the derivative to analyze a problem, and one example of how to use the FTC to take derivatives.

Example 4.1. For all rectangles with area R, the one with the smallest perimeter is the square.

Proof. So: this is just looking at minimizing the function $2x + 2\frac{R}{x}$. To do this, we differentiate:

$$\frac{d}{dx}(2x + 2\frac{R}{x}) = 2 - 2\frac{R}{x^2},$$

which is equal to 0 iff $x^2 = R$. So this gives us that $x = \sqrt{R}$ is either a minimum or maximum of the total perimeter. As the derivative is positive for $x > \sqrt{R}$ and negative for $x < \sqrt{R}$, we know that this is a minimum. □

Example 4.2. Let $f(x)$ be defined as

$$f(x) := \int_{x^3}^{x^2} \frac{t^6}{1+t^4} dt.$$

What is $f'(x)$?

Proof. First, define

$$F(x) := \int_0^x \frac{t^6}{1+t^4} dt.$$

Then, we have that $f'(x)$ is equal to (by the chain rule)

$$\frac{d}{dx} \int_{x^3}^{x^2} \frac{t^6}{1+t^4} dt = 2x \cdot F'(x^2) - 3x^2 \cdot F'(x^3).$$

By the Fundamental Theorem of Calculus, this is just

$$2x \cdot \frac{x^{12}}{1+x^8} - 3x^2 \cdot \frac{x^{18}}{1+x^{12}}.$$

□

Exercise 4.3. All previous exercises, plus: (1) Can you cover $\mathbb{R}^2$ with disjoint circles of positive radius? (2) In the plane $\mathbb{R}^2$, can you plant uncountably many disjoint T's (of nonzero width and height) on the x-axis? (Here, a T rooted at a with height b, width c is simply the union of the two sets $T_a := \{(a, x) : x \in [0, b]\}$ and $T^b := \{(x, b) : x \in [b - c, b + c]\}$. i.e. it "looks" like a T.)