INTEGRATION BY PARTS / TAYLOR SERIES / L'HÔPITAL'S RULE

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So: the homework for these past two weeks was fine overall. Minor errors with forgetting to show work or cite why you could move from one step to another were the main things that lost people points; actual comprehension of the material has been relatively high. Especially on the last homework; I expected that to go much worse than it did.

Also, several answers to the bonus questions were given over the last two weeks! We will review their answers at the end of this handout.

First, the "meat" of the last two weeks: (1) example calculations of integration by parts, (2) example calculations and applications of Taylor series in various situations, and (3) example calculations of limits via L'Hôpital's Rule.

1. Integration by Parts

Theorem 1.1. *The identity*

$$\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int cos^{n-2}(x)dx$$

holds for all $n \in \mathbb{N}$.

Proof. So, in the integral

$$\int \cos^n(x)dx,$$

set

(1.2)
$$U = \cos^{n-1}(x), \partial V = \cos(x)dx, \partial U = -(n-1)\cos^{n-2}(x)\sin(x)dx, V = \sin(x),$$

and apply integration by parts to get

$$\int \cos^n(x)dx = UV - \int V\partial U = \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\sin^2(x)dx$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-1}(x)(1-\cos^2(x))dx$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)dx - (n-1)\int \cos^n(x)dx,$$

which then implies that

$$\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)dx.$$

□

2. Taylor series

Theorem 2.1. *The Taylor polynomial of degree n of $\frac{1}{1+x}$ is*

$$T_n(\frac{1}{1+x}) = \sum_{i=0}^{n} (-1)^k \cdot x^k$$

Proof. So: notice that

$$(1+x) \cdot \left(\sum_{i=0}^{n} (-1)^k \cdot x^k \right) = \left(\sum_{i=0}^{n} (-1)^k \cdot x^k \right) + \left(\sum_{i=0}^{n} (-1)^k \cdot x^{k+1} \right)$$
$$= x^{n+1} \cdot (-1)^n + 1;$$

consequently, we can write $\frac{1}{1+x}$ as

$$\sum_{i=0}^{n} (-1)^k \cdot x^k + \frac{(-1 \cdot x)^{n+1}}{1+x}.$$

As the term on the right goes to 0 as x goes to 0, we have (by theorem 7.4 in Apostol) that

$$T_n(\frac{1}{1+x}) = \sum_{i=0}^{n} (-1)^k \cdot x^k.$$

□

Theorem 2.2. *The limit as $x \to 0$ of*

$$(1-x)^{1/x}$$

is e^{-1}.

Proof. So: recall that the Taylor series for $\ln(1-x)$ is

$$T_n(e^x) = \sum_{i=1}^{n} \frac{-x^i}{i}.$$

Then: we have that (by the above)

$$\lim_{x \to 0} (1-x)^{1/x} = \lim_{x \to 0} e^{ln(1-x)/x}$$

$$= \lim_{x \to 0} e^{\frac{-x+o(x)}{x}} = \lim_{x \to 0} e^{-1+\frac{o(x)}{x}} = e^{-1}.$$

□

3. L'Hôpital's Rule

Theorem 3.1. *The limit as $x \to 1$ of*

$$\frac{\ln(x) + 2x - 2}{e^x - xe}$$

is undefined.

Proof. So: because $\lim_{x \to 1} \ln(x) + 2x - 2 = 0$ and $\lim_{x \to 1} e^x - xe = 0$, we can use L'Hôpital's rule to get

$$\lim_{x \to 1} \frac{\ln(x) + 2x - 2}{e^x - xe} = \lim_{x \to 1} \frac{2 + 1/x}{e^x - e} = \lim_{x \to 1} \frac{3}{e^x - e},$$

which clearly diverges as $x\ to 1$. □

Theorem 3.2. *The limit as $x \to 0$ of*

$$\sin(x)^{\sin(x)}$$

is 1.

Proof. So: write

$$\lim_{x\to 1} \sin(x)^{\sin(x)} = \lim_{x\to 1} e^{\ln(\sin(x))\sin(x)} = \lim_{x\to 1} e^{\frac{\ln(\sin(x))}{1/\sin(x)}}$$

Because e is continuous and the limits as $x \to 0$ of the numerator and denominator in $\frac{\ln(\sin(x))}{1/\sin(x)}$ are 0, we can apply L'Hôpital's Rule to get

$$= \lim_{x\to 1} e^{\frac{cos(x)/(sin(x)}{cos(x)/\sin^2(x)}} = \lim_{x\to 0} e^{sin(x)} = 1.$$

□

Theorem 3.3. *The limit as $x \to \infty$ of*

$$\frac{a^x}{x^n}$$

is

$$\infty$$

for any $a > 1$ and $n \in \mathbb{N}$.

Proof. So: observe that we can apply L'Hôpital's rule to this limit, as both the numerator and denominator diverge as $x \to \infty$. So:

$$\lim_{x\to\infty} \frac{a^x}{x^n} =^{L'H} \lim_{x\to\infty} \frac{\ln(a)a^x}{nx^{n-1}} = \lim_{x\to\infty} \frac{\ln^2(a)a^x}{n(n-1)x^{n-2}} =$$

$$\lim_{x\to\infty} \frac{\ln^n(a)a^x}{n!} = \infty.$$

□

4. Bonus Exercises, Past

So: recall the following definition:

Definition 4.1. A n-**coloring** on $\mathbb{R}^2$ is an assignment of n distinct colors $a_1 \ldots a_n$ to the points $(x, y) \in mathbbR^2$, such that every point has precisely one color associated to it.

We asked two questions in previous classes:

Question 4.2. *Is there a 3-coloring on $\mathbb{R}^2$ such that for any two points $(x, y), (a, b) \in \mathbb{R}^2$, if (x, y) and (a, b) are distance 1 from each other?*

Question 4.3. *For which numbers n are there n-colorings on $\mathbb{R}^2$ such that for any two points $(x, y), (a, b) \in \mathbb{R}^2$, if (x, y) and (a, b) are distance 1 from each other?*

Solutions were given for the first question (a hint: construct a lattice made out of equilateral triangles all with side length 1. What are the colors of the vertices of the triangles? Why will this become a problem), and a bound of 8 (the optimal known bound is 7, I believe) was discovered for the second question (for 8: tile the plane with squares of side length .6, and assign every square a distinct color.) (7 can be found with a similar tiling approach.)