MATH 1A, SECTION 1, WEEK 1 - RECITATION NOTES

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Abstract. These are the notes from Thursday, Oct. 1's recitation on proofs. We try to define the mathematical concept of proof, and discuss some methods mathematicians use to create proofs.

1. Administrivia and Announcements

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- Recitation: 151 Sloan, 7:30-8:50 Th.
- Office: 360 Sloan
- Office Hours: Sunday, 9-10 pm / after recitation, for as long as questions exist.
- Homework Policy: Late homework – even late by only five minutes! – receives a 0, unless you have a written exemption from either the Deans or the student care center. This is a department-wide policy that I have absolutely no control over, so try to not get hit by this!

2. Random Question

Every week in recitation, I'll put up a question or two for people to think about if they've seen some of the material we're covering before and want something else to ponder. These are random, mostly mathematical puzzles I've ran into in my career as a student that I liked – if you're interested in any of them or happen to solve one of them, talk to me! I'm always happy to hear possible solutions or offer hints. Alternately, if you're not interested, don't worry; these are merely for the curious amongst you.

Question 2.1. *How many queens can you place on a 8×8 chessboard, such that no two can capture each other? How about a $n \times n$ chessboard? How many distinct queen-configurations can you come up with for a given board?*

3. What is a Proof?

In mathematics, a **proof** is a deductive argument that, from some collection of previously established truths, shows that some given statement – the thing that we seek to prove – is true. Essentially, writing a proof is like building a tower out of blocks; you start out with a collection of smaller established objects and "stack" them up to get a new object. A few examples will help to illustrate what we're talking about:

Proposition 3.1. *The sum of any two even numbers is even.*

Proof. Pick any two even numbers x and y. We seek to show that $x + y$ is also even.

So: where do we start? From the (very few) things we already know:

(1) the definition of an even number: i.e. because x is even, there must be an integer a such that $x = 2a$. Similarly, because y is even, there is an integer b such that $y = 2b$.
(2) the axioms of arithmetic: specifically, we know that
$$x + y = 2a + 2b = 2(a + b)$$
by the distributive axiom. Furthermore, we know that the term $a+b$ above is an integer, because the integers are closed under addition.
(3) the definition of an even number, again: from (2), we know that $x + y = 2(a + b)$, and is thus even by definition. So we're done!

□

The above proof was pretty ridiculously pedantic, but hopefully illustrates the basic idea we're going for here. Two more examples are below: in your HW this week, for questions 1 and 2 I'll be looking for a level of rigor at the level that we display in these two proofs.

Proposition 3.2. $0 \times r = 0$, *for any real number* r.

Proof. So: pick any real number r. Then, we have

$$\begin{aligned}
0 \times r &= (0 + 0) \times r && \text{(axiom of additive identity)}\\
\Rightarrow 0 \times r &= 0 \times r + 0 \times r && \text{(axiom of distributivity)}\\
\Rightarrow 0 \times r - 0 \times r &= 0 \times r + 0 \times r - 0 \times r && \text{(axiom of existence of additive inverses)}\\
\Rightarrow 0 &= 0 \times r && \text{(definition of additive inverses)}
\end{aligned}$$

□

Proposition 3.3. *Pick any two real numbers* x, y *such that* $x < y$. *Then there is a real number* z *such that* $x < z < y$.

Proof. So: set $z = \frac{1}{2} \cdot (y - x) + x$. Because the real numbers are closed under multiplication and addition, we know that z is a real number. We claim that this z is the number we are looking for: i.e. that $x < z < y$.

By definition, we know that $x < z$ holds if and only if $z - x$ is a positive number; but
$$z - x = \frac{1}{2} \cdot (y - x) + x - x = \frac{1}{2} \cdot (y - x)$$
by our definition of additive inverses. As well, we know from our axioms that

- $\frac{1}{2}$ is positive,
- $(y - x)$ is positive, because $y > x$, and
- the product of two positive numbers is positive.

So, we have that $z - x = \frac{1}{2} \cdot (y - x)$ is positive, and thus that $x < z$.

Similarly, we know that $y > z$ holds if and only if $y - z$ is positive: so, calculating

$$\begin{aligned} y - z &= y - \left(\frac{1}{2} \cdot (y - x) + x\right) \\ &= y - \left(\frac{y - x + 2x}{2}\right) \\ &= y - \frac{y + x}{2} \\ &= y + \frac{-y - x}{2} \\ &= \frac{2y - y - x}{2} = \frac{y - x}{2} = \frac{1}{2} \cdot (y - x) \end{aligned}$$

which is positive (as we showed earlier.) So $y > z$.

Combining, we have that $x < z < y$, as we sought to prove. □

So, these are pretty much what proofs are like! Notice how we avoided the following two pitfalls in the above proofs:

(1) **Using empirical reasoning.** If you ask the average person to prove something like "The sum of two even numbers are even," they'll often say something like "That's just how they work! Like, $8 + 8 = 16, 4 + 32 = 36$." This is **not a proof**! I cannot reiterate this enough – **examples do not prove things.** They give you intuition, can help you get a feel for what's going on, and can disprove certain things – but they do not prove a statement.

(2) **Not using words.** A common misconception of mathematics is that it just consists of long strings of equations occasionally joined by logical constructions; fifteen-page integrals and the such. Thankfully, that's completely rubbish. Mathematical proofs are arguments, and as such use lots and lots of words! Paragraphs of words, explaining to the reader what's going on and what the author is doing. In the proofs above, lots of verbiage is used to indicate to the reader what is going on – in your proofs, you should do the same! Words are your friends.

4. Proof Methods

So, the proofs we just did were pretty much straightforward. Sometimes, however, it can be easier to prove something using certain specific methods of proof: we outline two such methods below, and provide examples of their use.

4.1. **Proofs by Contradiction.**

Suppose that you want to show that some statement X is true. A proof by contradiction that X is true would go as follows:

- First, assume actually that X is false.
- Using this assumption that X is false along with your axioms and other things you've proven to be true, logically arrive at a contradiction – like "0=1" or something.
- Because the only thing you used that you were unsure about was the assumption that "X was false," you now know that that couldn't have held – i.e. that X is true!

An example is honestly the best way of properly illustrating what we're doing here:

Proposition 4.1. *No number is both even and odd.*

Proof. (By contradiction) Suppose not: that there is some number x such that x is both even and odd. By using the definitions of even and odd numbers, we then know that $x = 2a$ and $x = 2b + 1$ for some integers a and b.

By the transitive property of equality, we have then that

$$\begin{aligned}
2a &= 2b + 1 & \\
\Rightarrow 2a - 2b &= 2b + 1 - 2b & \text{axiom of existence of additive inverses} \\
\Rightarrow 2a - 2b &= 2b - 2b + 1 & \text{commutativity of addition} \\
\Rightarrow 2a - 2b &= 0 + 1 & \text{definition of additive inverses} \\
\Rightarrow 2a - 2b &= 1 & \text{definition of additive identity} \\
\Rightarrow 2(a - b) &= 1 & \text{axiom of distributivity} \\
\Rightarrow \frac{1}{2} \cdot 2(a - b) &= \frac{1}{2} \cdot 1 & \text{axiom of existence of multiplicative inverses} \\
\Rightarrow (a - b) &= \frac{1}{2} \cdot 1 & \text{definition of multiplicative inverses} \\
\Rightarrow (a - b) &= \frac{1}{2} & \text{definition of multiplicative identity}
\end{aligned}$$

by using our axioms – but this shows that $1/2$ is an integer, (as $a - b$ is an integer, because the integers are closed under addition.) This is clearly impossible, as the repeated addition and subtraction of 1 cannot yield a number between 0 and 1; so we have arrived at a contradiction. Thus no number is both even and odd. □

4.2. **Proofs by Induction.** Proofs by induction are used when you want to prove that some property holds for all of the natural numbers bigger than some number. (Usually, this number is 1.)

Explicitly, suppose that $P(n)$ is the property you're trying to show. (For example, $P(n)$ could be the statement that n is an integer.) To use induction to prove that $P(n)$ holds for every natural number $n \geq k$, we have to simply do two things:

1. Prove a **base case**: We start by showing that $P(n)$ holds for $n = k$. Again, usually this k is going to be 1; but occasionally you'll want to prove something that's true only for every number bigger than 2, say.
2. Prove the **inductive step**: Here, we assume that $P(n)$ is true, and we use that knowledge to prove that $P(n + 1)$ is true.

If we can do these two things, we've actually proven that our statement holds for all of the natural numbers bigger than the one we started on!

That this is a proof may seem odd at first; a good mental picture to help you understand what's going on might be to think of dominoes. Suppose you've lined up a bunch of dominoes, and you knock the first one over. What's happened to all of the other dominoes? They're all knocked over, too – because you (1) knocked over the first domino, and (2) set them up so that if the n-th domino fell, that it would knock over the $n + 1$-th domino. That's all that induction is; just knocking over dominoes.

If metaphors are not motivating, try these two examples instead to get a flavor of what's going on here:

Proposition 4.2.

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2},$$

for all natural numbers n.

Proof. (By induction.) Base case: let $n = 1$. Then the above is trivially true, as

$$\frac{1(1+1)}{2} = \frac{2}{2} = 1.$$

Inductive step: Assume that

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

We then seek to show that

$$1 + 2 + 3 + \ldots + n + n + 1 = \frac{(n+1)(n+2)}{2}.$$

To see this, simply regroup and apply the inductive hypothesis, as below:

$$\begin{aligned} 1 + 2 + 3 + \ldots + n + n + 1 &= (1 + 2 + 3 + \ldots + n) + n + 1 \\ &= \frac{n(n+1)}{2} + n + 1 \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2}. \end{aligned}$$

This finishes the inductive step, and thus the entire proof. □

Proposition 4.3.

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{n}\right) = \frac{1}{n},$$

for all $n \geq 2$.

Proof. (By induction.) Base case: Note here that we start our induction at 2, not 1! This is because the above formula doesn't quite make sense for values of n smaller than 2.

So: this is trivial, as

$$\left(1 - \frac{1}{2}\right) = 1/2.$$

Inductive step: assume that

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{n}\right) = \frac{1}{n}.$$

We then seek to show that this holds for $n + 1$ – i.e. that

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n+1}\right) = \frac{1}{n+1}.$$

To see this, note again that by simply regrouping and applying our inductive hypothesis, we have

$$\begin{aligned}\left(1-\frac{1}{2}\right)\cdots\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n+1}\right) &= \left(\left(1-\frac{1}{2}\right)\cdots\left(1-\frac{1}{n}\right)\right)\left(1-\frac{1}{n+1}\right)\\ &= \left(\frac{1}{n}\right)\left(1-\frac{1}{n+1}\right)\\ &= \left(\frac{1}{n}\right)\left(\frac{n}{n+1}\right)\\ &= \frac{1}{n+1}.\end{aligned}$$

This finishes the inductive step, and thus the proof. □

On your HW this week, we're looking for your inductive proofs to look like the ones above – take the time to clearly identify your base cases and inductive steps, and describe in words what you're doing and when you use your inductive hypothesis. As always, write me if you're unsure about one of your proofs.