MATH 1A, SECTION 1, WEEK 2 - RECITATION NOTES

TA: PADRAIC BARTLETT

Abstract. These are the notes from Thursday, Oct. 8's recitation on rational numbers, irrational numbers, and the infinte. We start by defining the rational numbers as equivalence classes (and define just what an equivalence class and an equivalence relation are along the way.) From there, we formally define functions and the notions of injective, surjective, and bijective functions, and use these notions to talk about the "sizes" of sets – even infinite ones! In particular, we close by showing that, in a very real way, that there are "more" real numbers than there are rational numbers.

1. Administrivia and Announcements

- Sickness: so, there's apparently a lot of various plagues running around! If you're sick and can't complete the set on time, go to the SCC as soon as you're able to get a note, and get it to me (ideally by putting it in my mailbox.)
- Late sets: also, turn these into my mailbox in Sloan, and not the normal HW box.

2. HW comments

- Average: 60%, with a completely flat distribution.
- Common problems: just a lot of issues with getting used to proofs: notational issues, people not using their words to describe what they're doing, proofs being done "backwards" (i.e. students were assuming at the start of their proof what they wanted to show,) long strings of equations without any logical connectives ($\Leftrightarrow, \Rightarrow$) to show the connections between them. It'll get easier with practice.

3. Defining the Rational Numbers - Equivalence Relations

So: first, recall the definition of the rational numbers that was put forth in Apostol, and that we had to use on the last set:

Definition 3.1. Quotients of integers a/b are called **rational numbers.**

This definition, as you may remember from your first problem set, was kind of problematic – specifically, it wasn't clear that for any rational number a/b we could find a rational number c/d such that $a/b = c/d$ and $GCD(c, d) = 1$. As well, from the definition it's not entirely clear which quotients of integers are going to represent *different* rational numbers: i.e. while we know that $1/2$ and $2/4$ are the same number, it's not immediately obvious from the definition. We'd like a new definition that fixes these problems: to do that, we have to introduce some definitions

Definition 3.2. A **relation** R on a set A is a collection of ordered pairs of elements of A. For any two elements $x, y \in A$, we'll write xRy whenever the pair (x, y) is in our relation.

Examples 3.3. The symbol “$<$” is a relation on the real numbers. Explicitly, it consists of all of the pairs (x, y) such that $y - x > 0$.

The phrase “is a student of” is a relation on the set of people at Caltech. Explicitly, it consists of all of the pairs of people (x, y) such that x is a student of y.

Definition 3.4. An **equivalence relation** $\sim$ on a set A is a relation that satisfies the following properties:

- **Reflexivity**: for every $x \in A$, $x \sim x$.
- **Symmetry**: for every $x, y \in A$, if $x \sim y$, $y \sim x$
- **Transitivity**: for every $x, y, z \in A$, if $x \sim y$ and $y \sim z$, $x \sim z$.

If we have an equivalence relation on a set A, we can then define the **equivalence class** of an element x in A as the collection of all things that are related to x under the relation $\sim$.

Examples 3.5. The relation “$<\leq$” is not an equivalence relation on the real numbers: while it is indeed transitive (as $x \leq y \leq z$ implies $x \leq z$) and reflexive (as $x \leq x, \forall x$), it's not symmetric (because $x \leq y$ does **not** imply $y \leq x$.)

Similarly, the relation “is a sibling of” on the set of people in LA is not an equivalence relation – while this is symmetric and transitive, it's decidedly not reflexive (as most people are not their own brother or sister.)

The relation “$=$” on the real numbers, however, is an equivalence relation, as it's obviously symmetric, reflexive and transitive. The equivalence class of any element $x \in \mathbb{R}$ is just x, as x is the only real number equal to x.

Similarly, the relation “has the same hair color as” defines an equivalence relation on the people in this class – it's clearly a reflexive relation (as one always has the same hair color as oneself,) symmetric (as if x has the same hair color as y, then y trivially has the same hair color as x,) and transitive (as if x has the same hair color as y, who has the same hair color as z, then they all have the same hair color.) The equivalence class of any person in our class is thus the collection of all of the people with the same hair color as that person – in other words, our equivalence relation is just a way of dividing our class into smaller sets, each of which is made of people all with the same hair color!

In general, equivalence classes are just generalizations of the notion of equality – ways of grouping things together based on certain properties, like hair color, or being the same number, or the same height, or from the same state. They offer us a way to say that two things, as far as we are concerned, are mathematically the “same,” even though they outwardly may look different.

It turns out that we can define the rational numbers as an equivalence relation! This will give us a rigorous way to insist that numbers like $1/2$ and $2/4$ are representing the same things.

Definition 3.6. Take the set of all pairs of integers a/b such that $b > 0$. We define a relation $\sim$ on this set as follows:

$$a/b \sim c/d \Leftrightarrow ad = bc.$$

Proposition 3.7. *The relation $\sim$ defined above is an equivalence relation.*

Proof. To see this, we just have to check the three defining properties of an equivalence relation:

- Reflexivity: we want to show that for any pair $a/b, b \neq 0$, that $a/b \sim a/b$. But
$$a/b \sim a/b \Leftrightarrow ab = ab$$
is obviously true; so we're done.
- Symmetry: we want to show that for any two pairs $a/b, c/d, b, d \neq 0$, that $a/b \sim c/d$ implies $c/d \sim a/b$. But
$$a/b \sim c/d \Leftrightarrow ad = bc \Leftrightarrow bc = ad \Leftrightarrow c/d \sim a/b,$$
so we're done.
- Transitivity: we want to show that for any three pairs $a/b, c/d, e/f, b, d, f \neq 0$, that if $a/b \sim c/d$ and $c/d \sim e/f$, we have that $a/b \sim e/f$. But
$$\begin{aligned} a/b \sim c/d &\Leftrightarrow ad = bc \Leftrightarrow a = bc/d \Leftrightarrow af = bcf/d \\ c/d \sim e/f &\Leftrightarrow cf = ed \Leftrightarrow e = cf/d \Leftrightarrow eb = bcf/d \\ &\Rightarrow af = bcf/d = eb \Leftrightarrow a/b \sim e/f. \end{aligned}$$

  So we're done!

This proves that $\sim$ is an equivalence relation on our set of all pairs a/b where $b > 0$. □

So: with this in hand, we can finally define the rational numbers:

Definition 3.8. Take the set A of all pairs of integers a/b where $b > 0$. We define the rational numbers to be the collection of all of the equivalence classes of the set A under the equivalence relation $\sim$ we just defined above.

In other words: a rational number $r \in \mathbb{Q}$ is the **set** of all of the fractions that are equal to r! i.e.
$$\{1/2, 2/4, 3/6, 4/8, 5/10, 6/12, 7/14 \ldots\}$$
is the rational number "1/2" – though we could just as easily call it "2/4," or even "24/48."

In general, we won't ever make use of this crazy infinite-set way of describing the rational numbers – we'll just think of them as pairs a/b where $b \neq 0$. But the cool thing about this definition is it fixes our two earlier problems:

(1) under the definition above, the numbers 1/2 and 2/4 are representing the same set –so we can actually think of them as being equal! This is a major advantage over our original definition, which did not make it clear that these kinds of fractions should be thought of as equal to each other.

(2) Given any rational number r, we can now find integers a, b such that $a/b = r$ and a, b have no common factors! This again comes from our above definition.

To see this: take any rational number r, and look at the set of fractions represented by r. Look, specifically, at the denominators of all of those fractions – this is a collection of positive integers, and thus has a smallest member b. Let a be the numerator corresponding to this denominator b, and look at a/b – we claim that a and b have no common factors.

To see this, we proceed by contradiction: suppose that they had some common factor c. Then look at $(a/c)/(b/c)$ – by definition, r is a representative of this fraction as well, and it has a smaller denominator than a/b did. This contradicts our choice of b (because we picked the smallest denominator!) – so we know that no such common factor could have existed.

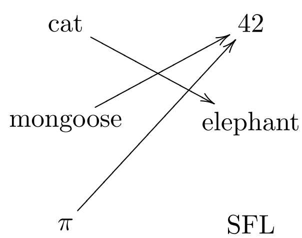
4. Functions - A Formal Viewpoint

Definition 4.1. A **function** f with domain A and range B is a collection of pairs $(a, b), a \in A, b \in B$ such that there is exactly one pair (a, b) for every $a \in A$.

Example 4.2. $f(x) = \sin(x)$, with domain $\mathbb{R}$ and range $\mathbb{R}$, is a function.

$g(x) = \sin(x)$, with domain $[0, \infty)$ and range $\mathbb{R}$, is also a function. g is in fact a different function than the function f we defined above, because it has a different domain.

The function h depicted below by the three arrows is a function, with domain {cat, mongoose, π} and range {42, elephant, SFL}.



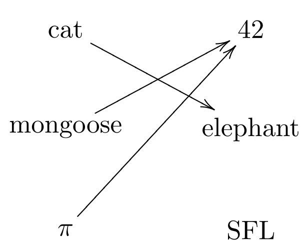
This may seem like a silly example, but it is important to remember this idea of functions merely being maps between sets. Often, students fall into the trap of assuming that everything's a map from $\mathbb{R} \to \mathbb{R}$, and furthermore that it will have some nice form, like x^2 or $\arcsin(x)$; in "reality," however, your typical function will not have a nice closed form, nor will it necessarily be a map on something as well-behaved as $\mathbb{R}$.

Definition 4.3. We call a function f **injective** if it never hits the same point twice – i.e. for every $b \in B$, there is **at most one** $a \in A$ such that $f(a) = b$.

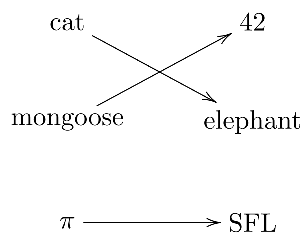
Example 4.4. The function $f(x) = x^2$ with domain $\mathbb{R}$ and range $\mathbb{R}$ is not injective, because it sends -1 and 1 to the same place (as $-1^2 = 1^2 = 1$.)

However, $g(x) = x^2$ with domain $[0, \infty)$ and range $\mathbb{R}$ is an injective function! This is because any two different positive numbers have different squares.

The function h depicted below by the three arrows is not injective, as both the mongoose and π are sent to 42:



However, the function i depicted below is injective, as it sends each element in its domain to different places:

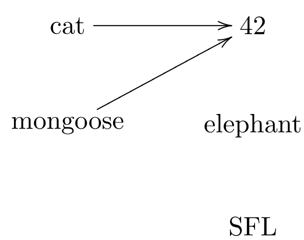


Definition 4.5. We call a function f **surjective** if it hits every single point in its range – i.e. if for every $b \in B$, there is **at least one** $a \in A$ such that $f(a) = b$.

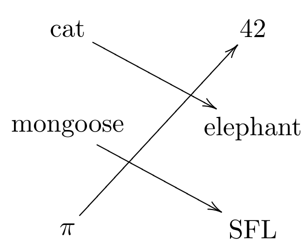
Example 4.6. The function $f(x) = x^2$ with domain $\mathbb{R}$ and range $\mathbb{R}$ is not surjective, because it never hits any of the negative numbers.

However, $g(x) = x^2$ with domain $\mathbb{R}$ and range $[0, \infty)$ is a surjective function! This is because any positive number has a square root.

The function h depicted below is not surjective, as nothing hits the element SFL:



However, the function i depicted below is surjective, as it hits every element in the range:

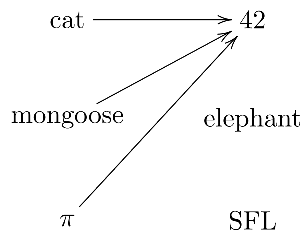


Definition 4.7. We call a function **bijective** if it is both injective and surjective.

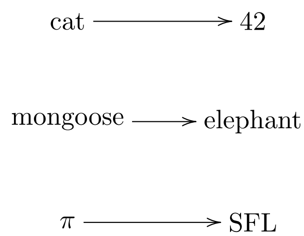
Example 4.8. The function $f(x) = x^2$ with domain $\mathbb{R}$ and range $\mathbb{R}$ is not bijective because it isn't surjective.

However, $g(x) = x^2$ with domain $[0, \infty)$ and range $[0, \infty)$ is a bijective function! This is because any positive number has exactly 1 square root, which is also positive.

The function h depicted below by the three arrows is not bijective, as nothing hits the element SFL (thus making it not surjective) and the element 42 is hit three times (thus making it not injective):



However, the function i depicted below is bijective, as it hits every element in the range exactly once:



5. Sizes of Infinity

Definition 5.1. We say that two sets A, B are the "same size" – formally, that they are sets of the same **cardinality** – if there is a bijection from A to B.

Remark 5.2. It's worth noting that in the finite case, this makes complete sense. Why? If A and B are finite sets, it's pretty clear that we can only have an injective function from A to B if there aren't more elements in A than there are spaces for them in B. Also, we're only going to be able to have a surjective function from A to B if there aren't more elements in B than there are elements in A to hit them with; combining these two results, we see that A and B would have to be the same size if there was a bijection between A and B.

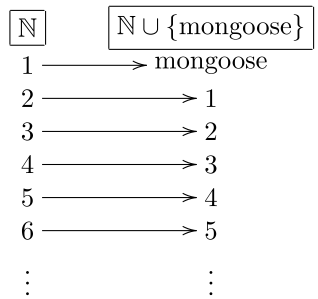
Definition 5.3. For any two sets A, B, we define the following sets:

- $A \cup B$: this is the set consisting of all of the elements in either A or B. Formally,
$$A \cup B = \{x | x \in A \text{ or } x \in B\}.$$
- $A \cap B$: this is the set of all of the elements in both A and B. Formally,
$$A \cap B = \{x | x \in A \text{ and } x \in B\}.$$
- $A \setminus B$: this is the set of all of the elements in A that are not also members of B. Formally,
$$A \cup B = \{x | x \in A \text{ and } x \notin B\}$$

So: with these definitions in hand, we set out to prove some surprising results.

Proposition 5.4. *The sets $\mathbb{N}$ and $\mathbb{N} \cup \{mongoose\}$ are the same cardinality.*

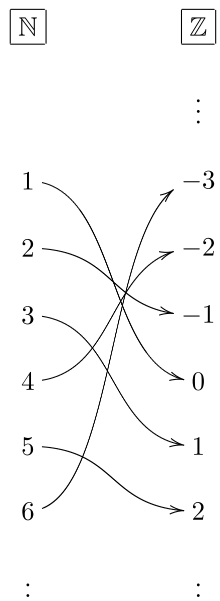
Proof. Consider the following map:



i.e. the map which sends 1 to the mongoose and sends $n \to n-1$, for all $n \geq 2$. This is clearly a bijection; so these sets are the same size. □

Proposition 5.5. *The sets* $\mathbb{N}$ *and* $\mathbb{Z}$ *are the same cardinality.*

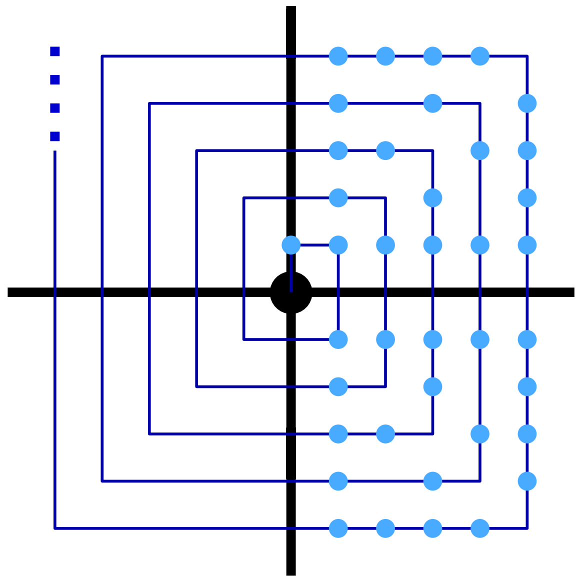
Proof. Consider the following map:



i.e. the map which sends $n \to (n-1)/2$ if n is odd, and $n \to -n/2$ if n is even. This, again, is clearly a bijection; so these sets are the same cardinality. □

Proposition 5.6. *The sets $\mathbb{N}$ and $\mathbb{Q}$ are the same cardinality.*

Proof. Consider the following picture:



It is left to the reader to understand how the above picture is related to a bijection from $\mathbb{N}$ to $\mathbb{Q}$. □

Proposition 5.7. *The sets $\mathbb{N}$ and $[0,1)$ are in fact different cardinalities – i.e. one of them is "larger" than the other.*

Proof. (This is Cantor's famous diagonalization argument.) Suppose not – that they were the same cardinalities. Pick a bijection f from $\mathbb{N}$ to $[0,1)$.

For every $n \in \mathbb{N}$, look at the number $f(n) \in [0,1)$. This is a number between 0 and 1; so it has a decimal representation. Pick numbers $a_{n,1}, a_{n,2}, a_{n,3}, \ldots$ that correspond to this decimal representation – i.e. pick numbers $a_{n,i}$ such that

$$f(n) = .a_{n,1}a_{n,2}a_{n,3}\ldots$$

For example, if $f(4) = .125$, we would pick $a_{4,1} = 1, a_{4,2} = 2, a_{4,3} = 5$, and $0 = a_{4,4} = a_{4,5} = a_{4,6} = \ldots$, because the first three digits are 1,2, and 5, and the rest of them are zeroes.

Now, write these numbers in a table, as below:

$f(1)$	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	$a_{1,4}$	$\ldots$
$f(2)$	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	$a_{2,4}$	
$f(3)$	$a_{3,1}$	$a_{3,2}$	$a_{3,3}$	$a_{3,4}$	
$f(4)$	$a_{4,1}$	$a_{4,2}$	$a_{4,3}$	$a_{4,4}$	
$\vdots$	$\vdots$				$\ddots$

In particular, look at the entries $a_{1,1}a_{2,2}a_{3,3}\ldots$ on the diagonal. We define a number B using these digits as follows:

- Define $b_i = 2$ if $a_{i,i} \neq 2$, and $b_i = 8$ if $a_{i,i} = 2$.

- Define B to the be the number with digits given by the b_i – i.e.

$$B = .b_1 b_2 b_3 b_4 \dots$$

This generates a number B that's just a string of 2's and 8's, and lies in $[0, 1)$. So, if our function f is indeed a bijection, it must be in particular a surjection and thus there must be some k such that $f(k) = B$. But the k-th digit of $f(k)$ is $a_{k,k}$ by construction, and the k-th digit of B is b_k – these are different numbers (by construction,) so $f(k)$ cannot equal B! So f cannot be a bijection, and thus these sets are of different sizes.

□