MATH 1A, SECTION 1, WEEK 7 - RECITATION NOTES

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Abstract. These are the notes from Thursday, Nov. 12th's recitation, on the relationship between integration and differentiation. We focus on the fundamental theorems of calculus, integration by substitution, and integration by parts, calculating several examples along the way.

1. Homework comments

- Homework average: around 78%.
- Comments: not really anything in specific. People, as always, could explain what they're doing in their HW a little better: explicitly, most people should be using more words and rereading their proofs for errors. But there was no terribly "big" idea that caught everyone up – the HW average this week was 78% largely because most people just forgot to do a part of a problem somewhere in the set, not because of conceptual errors.

2. The Fundamental Theorems of Calculus

So: the phrase "Fundamental Theorem of Calculus" certainly *sounds* intimidating – as though it is something of the "final boss" of calculus theorems, and thus must be insanely complicated and difficult to use. Thankfully, that's complete bunk; the fundamental theorems of calculus are simply called those because they're really basic results which we use all the time. They're pretty simple results, which we restate below for your reading convenience:

Theorem 2.1. *Let $[a, b]$ be some interval. If f is an integrable function over the interval $[a, x]$ for any $x \in [a, b]$ and $c \in [a, b]$, then the function*

$$A(x) := \int_c^x f(t)dt$$

exists for all $x \in [a, b]$ and has derivative $A'(x) = f(x)$.

Theorem 2.2. *Let $[a, b]$ be some interval. Suppose that $f(x)$ is a function that has $P(x)$ as its* ***primitive*** *– i.e. $f(x)$ is the derivative of $P(x)$. Then, we have that*

$$P(x) = P(c) + \int_c^x f(x)dx,$$

for all $x, c \in [a, b]$.

In other words, these theorems tell us that integration and differentiation are (in some appropriate sense) "inverses" to each other! – i.e. that if we integrate and then differentiate, or differentiate and then integrate, we're getting the "same" function back (up to a constant.)

We illustrate the use of these theorems with a pair of quick examples:

Example 2.3. Calculate the derivative of the function

$$F(x) = \int_0^{x^2} \sin(t)dt.$$

Proof. First, define the function $G(x)$ as

$$G(x) := \int_0^x \sin(t)dt.$$

By the fundamental theorem of calculus, we know that

$$G'(x) := \sin(x).$$

Thus, because $G(x^2) = F(x)$, we can just use the chain rule to see that

$$\begin{aligned}(F(x))' &= (G(x^2))' \\ &= 2x \cdot G'(x^2) \\ &= 2x \cdot \left(\int_0^x \sin(t)dt\right)' \\ &= 2x \cdot \sin(x).\end{aligned}$$

□

Example 2.4. Calculate the derivative of the function

$$F(x) = \int_{1/x}^x \frac{1}{t}dt,$$

whenever $t > 0$.

Proof. First, define the function $G(x)$ as

$$G(x) := \int_1^x \frac{1}{t}dt.$$

Then, by the fundamental theorem of calculus, we have that

$$G'(x) := 1/x.$$

So: note that

$$F(x) = \int_{1/x}^x \frac{1}{t}dt = \int_1^x \frac{1}{t}dt - \int_1^{1/x} \frac{1}{t}dt = G(x) - G(1/x).$$

(Note that we defined the function G here as an integral starting at 1, not 0! This is because the integral $\int_0^x \frac{1}{t}dt$ doesn't even exist whenever x is nonzero. So, when you use linearity of your integrals to split them apart, do be careful that you're not accidentally breaking your integral into parts that don't exist!)

Then, with this expression of $F(x) = G(x) - G(1/x)$, we can just proceed by the chain rule:

$$\begin{aligned}(F(x))' &= (G(x) - G(1/x))' \\ &= G'(x) - (-\frac{1}{x^2}) \cdot G'(1/x) \\ &= 1/x + \frac{1}{x^2} \cdot \frac{1}{1/x} \\ &= 2/x.\end{aligned}$$

□

3. Integration by Substitution

So: what is integration by substitution? Formally, it's the statement that (for f integrable and g integrable and differentiable,)

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(x)dx.$$

(You can think of this as basically saying that "the chain rule works in reverse.")

Integration by substitution is a really useful technique – it allows us to calculate integrals that on first glance look really hard, by simply replacing complicated expressions with very simple ones. It's really easier to see how this works/why it works with some examples: so we offer a few below.

Example 3.1. What is

$$\int_0^2 x^2 \sin(x^3)dx \ ?$$

Proof. So: how do we calculate such an integral? Direct methods seem unpromising, and using trig identities seems completely insane. So we'll try a substitution!

What should we pick? This is the only "hard" part about integration by substitution – making the right choice on what to substitute in. In most cases, what you want to do is to find the part of the integral that you don't know how to deal with – i.e. some sort of "obstruction." Then, try to make a substitution that (1) will remove that obstruction, usually such that (2) the derivative of this substitution is somewhere in your formula.

Here, for example, the term $\sin(x^3)$ is definitely an "obstruction" – we haven't developed any techniques for how to directly integrate such things. So, we make a substitution to make this simpler! In specific: Let $u = x^3$. This turns our term $\sin(x^3)$ into a $\sin(u)$, which is much easier to deal with Also, the derivative $du = 3x^2dx$ is (up to a constant) being multiplied by our original formula – so this substitution seems quite promising. In fact, if we calculate, we have that

$$\int_0^2 x^2 \sin(x^3)dx = \int_0^2 \sin(x^3) \cdot \frac{1}{3} \cdot 3x^2dx = \int_0^8 \sin(u) \cdot \frac{1}{3} \cdot du$$

which is an integral we *can* calculate (it's $\frac{\sin(8)}{3}$.)

(Note that when we made our substitution, we also changed the bounds! Please, please, always change your bounds when you make a substitution!) □

Example 3.2. What is
$$\int_0^2 te^{-t^2/2}dt\ ?$$

Proof. Again, we look here for an "obstruction" that we can get rid of. The most obvious example is $e^{t^2/2}$, which (because it's a Gaussian function) is known to not have any remotely nice antiderivatives.

So: what could we try? Well, if we pick $u = -t^2/2$, we have that $du = -tdt$ definitely exists as a factor of our integrand – so this seems promising! Indeed,
$$\int_0^2 te^{-t^2/2}dt = \int_0^{-2} e^u \cdot (-1)du = \int_{-2}^0 e^u du,$$
which we can calculate easily (it's $1 - e^{-2}$.) □

Example 3.3. Calculate
$$\int_0^1 \sqrt{1-x^2}dx.$$

Proof. So: here, our glaring obstruction is the term $\sqrt{1-x^2}$. We could try using a substitution of the form $u = 1 - x^2$, or $u = x^2$ – but those aren't really going to help, because we don't have any $2x$'s running around in our formula. Really, pretty much any $u = f(x)$ style-substitution we make will run into this problem, as the expression $\sqrt{1-x^2}$ can't really be written as the product of anything!

So: what do we do? Well – if we can't take a function *out* of the equation (i.e. a $u = f(x)$ substitution,) maybe we should try putting a function *into* the equation! – i.e make a substitution of the form $x = f(u), dx = f'(u)du$. In specific, because the term is $\sqrt{1-x^2}$ – i.e. the function whose graph is the upper-half circle – we should be looking for a trig substitution.

Explicitly: when we say "trig substitution," what do we mean? Instead of defining some u in terms of x , define x as some function of u! – i.e. set $x = \sin(u)$ here. Then $dx = \cos(u)du$, and we have that
$$\begin{aligned}\int_0^1 \sqrt{1-x^2}dx &= \int_0^{\pi/2} \sqrt{1-\sin^2(u)}\cos(u)du\\ &= \int_0^{\pi/2} \cos^2(u)du.\end{aligned}$$

This is a definite improvement – but $\cos^2(u)$ doesn't look like something that will fall to substitution either! We are thus forced to switch tacks and discuss our next method of integration: □

4. Integration by Parts

So: where integration by substitution can be thought of as "the chain rule works in reverse," integration by parts can be thought of as "the product rule in reverse." Explicitly, it's the statement that for two functions $u(x), v(x)$ of x,
$$u(x)v(x) = \int u(x)\frac{dv}{dx}(x)dx + \int v(x)\frac{du}{dx}(x)dx.$$

Usually, people rearrange the terms and just write u, v for $u(x), v(x)$ – in this form, the equation is

$$\int u\ dv = uv - \int v\ du$$

Again, the "how to use this / why we use this" questions are best illustrated with a series of examples, the first of which we'll take from our last example in the integration-by-substitution section:

Example 4.1. What's

$$\int_0^{\pi/2} \cos^2(t)dt\ ?$$

Proof. So: the way that these kinds of proofs work is that we look at the quantity we're integrating (in this case, $\cos^2(t)$,) and try to divide it into two parts – a "u"-part and a "dv" part – such that when we apply the relation $\int u\ dv = uv - \int v\ du$, our expression gets simpler!

To ensure that our expression does in fact get simpler, we want to select our u and dv such that

(1) we can calculate the derivative du of u and the integral v of dv, and either
(2) the derivative du of u is **simpler** than the expression u, or
(3) the integral v of dv is **simpler** than the expression dv.

If we can always do this, then after a few applications of integration-by-parts the integral will be hopefully be simple enough that we can directly calculate it.

So: in the integral $\int \cos^2(t)$, we've only got three real options:

- Set $u = \cos^2(t), dv = dt$. This doesn't seem like it's going to get any simpler, because the integral of dv will add a t-term into the mix, and the derivative of u will continue to be a product of trig functions that we can't do much with.
- Set $u = 1, dv = \cos^2(t)dt$ This seems worse, as we can't even calculate what v would be!
- Set $u = \cos(t), dv = \cos(t)dt$. This, at least, doesn't seem to make anything worse – so let's try it!

So, if these are our substitutions, we then have that

$$\begin{array}{cc} u = \cos(t) & du = -\sin(t)dt \\ dv = \cos(t)dt & v = \sin(t)dt \end{array}$$

and thus that

$$\begin{aligned}\int \cos^2(t)dt &= uv - \int v\ du \\ &= \cos(t)\sin(t) - \int (\sin(t))(-\sin(t))dt \\ &= \cos(t)\sin(t) + \int (\sin^2(t)dt \\ &= \cos(t)\sin(t) + \int (1-\cos^2(t))dt \\ &= \cos(t)\sin(t) + t - \int \cos^2(t)dt\end{aligned}$$

which means that if we add $\int \cos^2(t)dt$ to both sides, we have

$$\begin{aligned}2\int \cos^2(t)dt &= \cos(t)\sin(t) + t + C \\ \Rightarrow \int \cos^2(t)dt &= \frac{\cos(t)\sin(t)+t}{2} + C'\end{aligned}$$

(where C' is just some arbitrary constant of integration.) This means that

$$\begin{aligned}\int_0^{\pi/2} \cos^2(t)dt &= \left(\frac{\cos(t)\sin(t)+t}{2} + C'\right)\Bigg|_0^{\pi/2} \\ &= \frac{\cos(\pi/2)\sin(\pi/2)+\pi/2}{2} + C' - \frac{\cos(0)\sin(0)+0}{2} - C' \\ &= \frac{\pi}{4}.\end{aligned}$$

So we're done! □

Example 4.2. Calculate

$$\frac{\ln(x)}{x^2}dx.$$

Proof. Of the things in this integral, the $\ln(x)$ stands out in particular as something that has a simpler derivative ($(\ln(x))' = 1/x$) and a much more complicated integral (which you usually use integration by parts to derive.) So: let's try setting

$$\begin{array}{ll} u = \ln(x) & du = \frac{1}{x}dx \\ dv = \frac{1}{x^2}dx & v = -\frac{1}{x}\end{array}$$

Then, we have that

$$\begin{aligned}
\frac{\ln(x)}{x^2}dx &= uv - \int v\ du \\
&= -\frac{\ln(x)}{x} - \int -\frac{1}{x}\ cdot\frac{1}{x}dx \\
&= -\frac{\ln(x)}{x} + \int \frac{1}{x^2}dx \\
&= \frac{\ln(x)}{x} - \frac{1}{x} + C.
\end{aligned}$$

□

To illustrate this strategy of simplification just once more, consider this last example:

Example 4.3. What's

$$\int x^3 e^x dx\ ?$$

Proof. Note that the integral of e^x is a particularly easy thing to calculate, as it's just e^x – as well, taking derivatives of x^3 will make this part of our integral simpler over time. This motivates us to make the choices

$$\begin{array}{ll} u = x^3 & du = 3x^2 dx \\ dv = e^x dx & v = e^x, \end{array}$$

which then gives us that

$$\begin{aligned}
\int x^3 e^x dx &= uv - \int v\ du \\
&= x^3 e^x - \int 3x^2 e^x dx
\end{aligned}$$

.

To calculate this next integral, for the same reasons as above, make the substitutions

$$\begin{array}{ll} u = 3x^2 & du = 6x dx \\ dv = e^x dx & v = e^x, \end{array}$$

which then tell us that

$$\begin{aligned}
\int x^3 e^x dx &= x^3 e^x - \int 3x^2 e^x dx \\
&= x^3 e^x - \left(uv - \int v\ du \right) \\
&= x^3 e^x - \left(3x^2 e^x - \int 6x e^x dx \right).
\end{aligned}$$

To calculate this last integral, for the same reasons as before, make the substitutions

$$\begin{array}{ll} u = 6x & du = 6dx \\ dv = e^x dx & v = e^x, \end{array}$$

which finally yields

$$\begin{aligned}
\int x^3 e^x dx &= x^3 e^x - \left(3x^2 e^x - \int 6xe^x dx\right) \\
&= x^3 e^x - \left(3x^2 e^x - \left(uv - \int v\ du\right)\right) \\
&= x^3 e^x - \left(3x^2 e^x - \left(6xe^x - \int 6e^x dx\right)\right) \\
&= x^3 e^x - \big(3x^2 e^x - (6xe^x - 6e^x)\big) \\
&= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x.
\end{aligned}$$

$\square$