MIDTERM REVIEW TOPICS

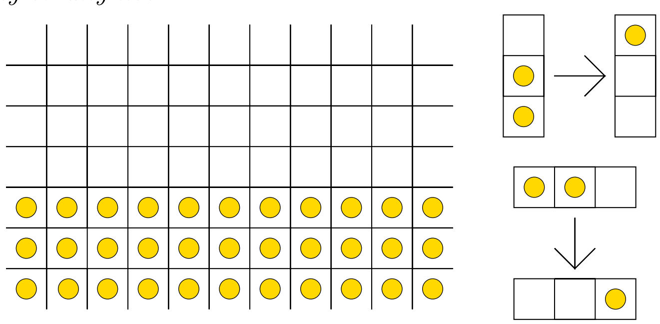
TA: PADRAIC BARTLETT - MATH 1B, WK. 6

1. Random Questions

Question 1.1. *So: Suppose you have a $\mathbb{Z} \times \mathbb{Z}$ grid of squares. Consider the following game we can play on this board:*

- *We start by putting one coin on every single square below the x-axis.*
- *If we have two coins in a row with an empty space ahead of them, we can "jump" one of the coins over the other, as depicted below.*

How "high" on the y-axis can you get a coin? Can you get one to height 3? Higher? Why or why not?



So: you appear to all have a midterm this week. Accordingly, these notes will be mostly review; the opening section is just a list of topics, which is followed by a series of worked example questions from Prof. Wilson's review sheet.

2. Overview of Topics

- Row-echelon form: what it is, how to transform matrices into row-echelon form (pivots!), what it tells you about the matrix (the rank of a matrix is the number of nonzero rows in its row-echelon form), when two matrices have the same row-echelon form (when they differ from each other by a series of elementary matrix operations).
- How to solve systems of linear equations with matrices – in both the homogenous case and the inhomogeneous case. (refer to the older notes if you forget how to do this, or to past HW, or your textbook, or come into office hours, or write me an email, or . . . yes. make sure you can do this!)
- Rank: how to determine it (by putting a matrix into row-echelon form), how it relates to linear independence (the rank of a matrix is the dimension of its row space – in other words, a matrix with exactly k linearly independent rows will have rank k and its row space will be of dimension k.)
- Orthogonality: what it means (a and b are orthogonal if $a \cdot b = 0$, where we understand $\cdot$ to be the dot product), and what you can do with it: i.e.

given a matrix U, you should be able to define $U^{\perp}$, the null space of U (i.e. the collection of all vectors perpindicular to U.) As well, you should know how to use orthogonality to find the area of a parallelogram, and how to turn a basis into an orthonormal basis via Gram-Schmidt! (We work an exercise on this later in the notes.)

- Linear independence and convex sets; know what they are (see the old notes if you forget), and be prepared to show that certain things are linearly independent/dependent/convex.
- Inverses: know how to find them, and when they exist (an inverse to a matrix A will exist precisely when the A is a $n \times n$ matrix of rank n. Note that, however, if A is not of this form, this doesn't mean that there isn't a matrix B such that $BA = I$; these can happen, and we call them either **left** or **right** inverses where they occur. However, such matrices will never be such that $AB = BA = I$ if A isn't a $n \times n$ matrix of rank n.)

3. Worked examples

Question 3.1. *Show that the rank of the matrix below is 7.*

$$A = \begin{pmatrix} 7 & 2 & 2 & 2 & 2 & 2 & 2 \\ 2 & 7 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 7 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 7 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 7 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 7 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 7 \end{pmatrix}$$

Proof. So: note that because the rank of a matrix is invariant under elementary row operations (i.e adding scalar multiples of rows to other rows), we have that the above matrix has the same rank as the matrix acquired by adding every row to the first row,

$$\begin{pmatrix} 19 & 19 & 19 & 19 & 19 & 19 & 19 \\ 2 & 7 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 7 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 7 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 7 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 7 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 7 \end{pmatrix}.$$

Dividing the first row by 19, we have

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 7 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 7 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 7 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 7 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 7 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 7 \end{pmatrix};$$

subtracting twice the first row from every other row then yields

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix};$$

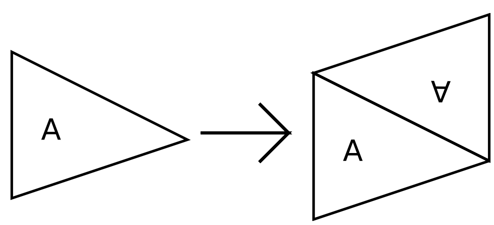
finally, dividing the second through sixth rows by 5 and subtracting them from the first yields

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

so we thus conclude that the rank of the identity matrix I_7 is the same as the rank of A; i.e. A has rank 7. □

Question 3.2. *What is the area of the triangle A in $\mathbb{R}^3$ with vertices $(1, 0, -1), (0, 1, 1)$, and $(1, 2, 3)$?*

Proof. So: pretend for a moment that the only thing you know about triangles is that sticking a pair of them together will give you a parallelogram, as depicted below:



Then, to find the area of the triangle A, it suffices to find the area of the parallelogram formed by combining two copies of the triangle A, and divide it by 2! So: to do this, first translate A by $(-1, 0, 1)$ to the triangle $(0, 0, 0), (-1, 1, 2)$, and $(0, 2, 4)$; this doesn't change the area of the original triangle. Then, we have that the parallelogram consisting of the two copies of A is spanned by $(0, 2, 4)$ and $(-1, 1, 2)$; so we proceed to find the area of this parallelogram via orthogonalization.

I.e.: we set

$$v_1 = (-1, 1, 2),$$

and by the Gram-Schmidt process set

$$v_2 = (0, 2, 4) - \frac{\langle (0, 2, 4), (-1, 1, 2) \rangle}{\langle (-1, 1, 2), (-1, 1, 2) \rangle} \cdot (-1, 1, 2) = (0, 2, 4) - (-\frac{10}{6}, \frac{10}{6}, \frac{20}{6}) = (\frac{5}{3}, \frac{1}{3}, \frac{2}{3}).$$

As a quick check to make sure we didn't do anything wrong, we note that

$$v_1 \cdot v_2 = (-1, 1, 2) \cdot (\frac{5}{3}, \frac{1}{3}, \frac{2}{3}) = 0.$$

Thus, we have that the area of the parellelogram spanned by $(-1, 1, 2)$ and $(0, 2, 4)$ is

$$|v_1||v_2| = \sqrt{1+1+4} \cdot \sqrt{\frac{25}{9} + \frac{1}{9} + \frac{4}{9}} = \sqrt{20} = 2\sqrt{5};$$

so the area of our triangle A is $\sqrt{5}$. □

Question 3.3. *Show that if* U *and* V *are subspaces of* $\mathbb{R}^n$ *and* $U \subseteq W$*, then* $\dim(U) \leq \dim(V)$.

Proof. So: Pick a basis $v_1 \dots v_k$ for U. This, by definition, is a linearly independent set of vectors in U that spans U.

Because $U \subseteq W$, we know that $v_1 \dots v_k \subset W$, and thus that W contains at least k linearly independent vectors. Then, two possibilities hold: either

(1) $W = \langle v_1 \dots v_k \rangle$. In this case, we know that because the $v_1 \dots v_k$ are linearly independent, that they are a basis for W, and thus that $\dim(W) = \dim(V)$.

(2) $W \supsetneq \langle v_1 \dots v_k \rangle$. In this case, we know that there is a vector $w_1 \in W$ such that $w_1 \notin \langle v_1 \dots v_k \rangle$; so, look at the set $\langle v_1 \dots v_k, w_1 \rangle$. Either this set is all of W or it is still a subset of W. If it is all of W, stop; otherwise, pick another vector w_2 and look at the set $\langle v_1 \dots v_k, w_1, w_2 \rangle$ …repeat this process until you eventually get a set $\langle v_1 \dots v_k, w_1 \ \dots w_l \rangle$ that spans W. This set is linearly independent, by construction (see the proof last week if you don't believe this,) spans W, and has more vectors in it than $\langle v_1 \dots v_k \rangle$.

So $\dim(W) \geq \dim(V)$.

□