PROJECTION AND DETERMINANTS

PADRAIC BARTLETT

1. Projection

Definition 1.1. So: given a pair of vectors v, u, we can define the **projection** of v onto u as the vector

$$proj_u(v) = \frac{\langle v, u\rangle}{\langle u, u\rangle} \cdot u;$$

we think of this as the "closest approximation" to v we can find in the space spanned by u.

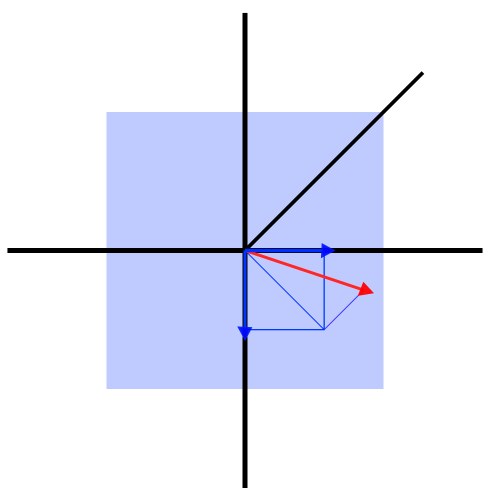
However, we often seek to talk about projection of vectors onto not merely other vectors, but spaces: here, we have the same general idea – we want to define the projection of v onto a space U as the "closest approximation" to v we can build with elements in U. (A potentially useful analogy is that of your three-dimensional body casting a 2-dimensional shadow – this is kind of like what projection does, sending things in some higher-dimensional space to an approximation of them in a lower-dimensional space.)

So: with that in mind, for a space U with an orthogonal basis $\{u_1 \ldots u_n\}$, we define the projection of a vector v onto U as

$$proj_U(v) = \sum_{i=1}^{n} proj_{u_i}(v);$$

i.e. the projection of v onto U is just the sum of the projections of v onto the various orthogonal components of U.

A picture of what's going on here is drawn below:



i.e. here you have the vector in red's projecton onto the space in blue, which decomposes into the projection of the red vector onto the two orthogonal axes which span the blue space.

2. Determinants

We have two analogous definitions of the determinant: we present both here, and try to expand on the second so that it is less mysterious.

Definition 2.1. For a $n \times n$ matrix $A = \{\alpha\}_{i,j=1}^{n}$, we can define the determinant of A recursively as follows:

- If A is a 1x1 matrix, we can define the determinant of A to simply be its one entry, α_{11}.
- If A is a $n \times n$ matrix, where $n \geq 1$, we can define the determinant of A by

$$\det(A) = \sum_{k=1}^{n} \alpha_{1k} \cdot \det(A_{1k}),$$

  where we denote the $n-1 \times n-1$ matrix A_{1k} as the matrix obtained from A by eliminating its first row and k-th column.

Definition 2.2. We also can define the determinant of $A = \{\alpha\}_{i,j=1}^{n}$ as the sum

$$det(A) = \sum_{\sigma \in S_n} (-1)^{sgn(\sigma)} \cdot \prod_{i=1}^{n} \alpha_{i,\sigma(i)}$$

So: to many of you, there are at least three undefined things in the definition above: (1) what σ is, (2) what S_n is, and (3) what $sgn(\sigma)$ is. So, let's define them!

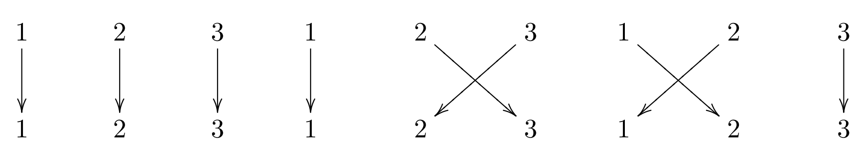
Definition 2.3. We define a **permutation** on a set S to be a map σ from S to itself that

1. "doesn't send any two elements to the same place" i.e. for any two distinct elements s_1, s_2 of S, $\sigma(s_1)$ is different from$\sigma(s_2)$.
2. "hits every element of S" – i.e. for any element s_2 in S, there is an element s_1 such that $\sigma(s_1) = s_2$.

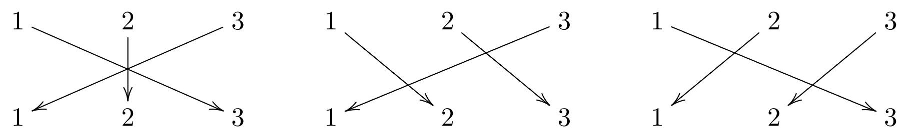
We think of permutations as just ways of rearranging the elements of S.

Definition 2.4. We define the sets S_n, for every n, as the collection of all permutations on the set of n elements $1, 2, 3\ ldotsn$.

Example 2.5. The set S_3 consists of the elements



id (23) (12)

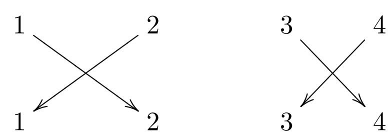


(13) (123) (132)

We denote these maps by the labels placed beneath them: i.e. (12) stands for the map which switches 1 and 2, and leaves 3 constant; (123) is the map which sends 1 to 2, 2 to 3, and 3 to 1; etc.

We will use this "cyclic" notation of denoting maps by $(abcd\ldots)$, as it's a lot quicker, and makes our last two definitions far easier.

Definition 2.6. We define a **cycle** as a permutation we can write as $(abcd\ldots)$ for some numbers a,b,c,d… All of the elements of S_3, for example, are cycles; however, there are permutations which are not cycles: take for example the following map in S_4,



(12)(34)

This map can't be written as a single cycle; it, however, can be written as a pair of cycles, (12)(34). In general, we can write all permutations in this form! This allows us to give our last definition,

Definition 2.7. For a permutation σ, we define $sgn(\sigma)$ as the number of elements in the cycles of σ minus the number of cycles that make up σ. i.e. – for $\sigma = (1234)(56)(432)$, we have that $sgn(\sigma) = 9-3 = 6$, as there are 9 numbers (counting repetition!) that are used in writing σ, and three cycles in which they are grouped.

So, this completely defines all of the terms in our new definition of the determinant! Next week: why we care.