EIGENVALUES AND PROBABILITY MATRICES

MATH 1B - NOTES, WK. 8

1. Last Week's HW

The average was about a 62; things were mostly solid, but the eigenspace/value question was a little shaky. As a result, I wanted to review a worked example of how to find eigenvalues for some of the notes here: if you're comfortable with this, feel free to skip the next section.

2. Eigenvalues

Question 2.1. *What are the eigenvalues and eigenspaces of the $n \times n$ matrix*

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \ddots & \\ 0 & \dots & & \dots & 0 & 1 \\ 1 & \dots & & \dots & 0 & 0 \end{pmatrix}?$$

So: we begin by calculating the characteristic polynomial of M, $\det(M - \lambda I)$: specifically, this is the determinant of the matrix

$$M - \lambda I = M' = \begin{pmatrix} -\lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & -\lambda & 1 & 0 & \dots & 0 \\ 0 & 0 & -\lambda & 1 & \dots & 0 \\ \vdots & & & \ddots & \ddots & \\ 0 & \dots & & \dots & -\lambda & 1 \\ 1 & \dots & & \dots & 0 & -\lambda \end{pmatrix}$$

So: how do we do this? First, recall that we can define the determinant **recursively** by expanding along a column of our matrix: i.e. that

$$\det(M') = \sum_{i=1}^{n} (-1)^{n-1} m_{i1} \det(M'_{i1}),$$

where $M = \{m_{ij}\}_{i,j=1}^{n}$, and M'_{ij} is the matrix formed by removing the j^{th} row and the i^{th} column from M'.

So: specifically, for our matrix M', we have that (because the first two entries in its leftmost column are the only nonzero entries) that

$$\begin{aligned}
\det(M') &= m_{11}\det(M_{11}) - m_{n1}\det(M_{n1})\\
&= -\lambda \cdot \begin{vmatrix} 1 & 0 & \dots & 0 \\ -\lambda & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & \dots & -\lambda & 1 \end{vmatrix} (-1)^{n-1} \cdot \begin{vmatrix} -\lambda & 1 & \dots & 0 \\ 0 & -\lambda & \dots & 0 \\ \vdots & & \ddots & \\ 0 & \dots & \dots & -\lambda \end{vmatrix}\\
&= (-\lambda)^n + (-1)^{n-1},
\end{aligned}$$

where the two determinant calculations of M_{11} and M_{n1} are made trivial by noting that they are upper-triangular and lower triangular matrices. (h/t: A. Craig)

So: we then have that this is equal to 0 whenever

$$(-\lambda)^n = (-1)^n;$$

specifically, when n is even we have that this holds whenever λ is a n-th root of unity (i.e. $\lambda = e^{2i\pi \cdot k/n}$) and whenever n is odd we have that this holds whenever λ is a n-th root of unity times $e^{2\pi i/2n}$.

So, we have then that the only real eigenvalues are

(1) ± 1, if n is even,
(2) 1, if n is odd.

To find the eigenspaces for these eigenvalues: note simply that because (for n even or odd)

$$M-(1)I = M' = \begin{pmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ 0 & 0 & -1 & 1 & \dots & 0 \\ \vdots & & & \ddots & \ddots & \\ 0 & \dots & & \dots & -1 & 1 \\ 1 & \dots & & \dots & 0 & -1 \end{pmatrix} \sim_R \begin{pmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ 0 & 0 & -1 & 1 & \dots & 0 \\ \vdots & & & \ddots & \ddots & \\ 0 & \dots & & \dots & -1 & 1 \\ 0 & \dots & & \dots & 0 & -0 \end{pmatrix}$$

by adding every row to the last row; as this is an upper-triangular matrix with a 0-row appended at its base, we can easily see that the rank of this matrix is $n-1$ and thus that the corresponding eigenspace for M is of rank 1, for n either odd or even.

As well, for n even, we have that

$$M-(-1)I = M' = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 \\ \vdots & & & \ddots & \ddots & \\ 0 & \dots & & \dots & 1 & 1 \\ 1 & \dots & & \dots & 0 & 1 \end{pmatrix} \sim_R \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 \\ \vdots & & & \ddots & \ddots & \\ 0 & \dots & & \dots & 1 & 1 \\ 0 & \dots & & \dots & 0 & -0 \end{pmatrix}$$

by alternately adding and subtracting rows from the last row (everything cancels because n is even!) As before, this matrix has rank $n-1$, and thus the eigenspace associated to -1 for n even has rank 1.

So in either case it suffices to find just one eigenvector, as it must consequently span the eigenspace. So, consider

$$x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ 1 \\ -1 \end{pmatrix}$$

Under M, x_1 gets taken to x_1 regardless of n being even, and if n is even x_2 gets taken to $-x_2$. So we've completely characterized the eigenvalues and spaces of this matrix! yay.

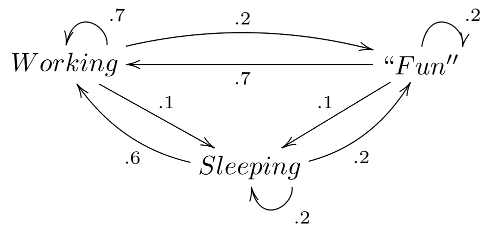
3. Probability Matrices

So: What are probability matrices?

Suppose you have a object (say, a student at Caltech,) and suppose you have a finite list of states the object can enter: say,

(1) Working
(2) Sleeping
(3) Building Giant Robots of Doom ("Fun").

Then suppose that objects move between these states with certain set probabilities: i.e. that a Caltech student who is asleep has, say, a %60 chance of starting work, a %20 chance of starting work on a giant robot, and a %20 chance of staying asleep.



We can then form a (left-stochastic, or column-stochastic) **probability matrix** P corresponding to these values, where we set p_{ij} to be the probability that a object in state j enters state i. (Note that this is backwards from the intuitive way in which you'd do this; i.e. I'd naively have thought that p_{ij} would be the probability that a object in state i enters state j. This would give you a row-stochastic/right-stochastic matrix, in which the rows would add to 1, as opposed to Wilson's definition, which ensures that the columns add to 1. Note also that your book defines probability matrices as row-stochastic matrices, so be careful in interpreting theorems from it and applying them to your HW.)

$$P = \begin{pmatrix} .7 & .6 & .7 \\ .1 & .2 & .1 \\ .2 & .2 & .2 \end{pmatrix}$$

In general, we can define a (left-stochastic) probability matrix as any matrix in which the columns all sum to 1 and all of the entries are ≥ 0, without making use of actual examples and probabilities – but the study and use of them is interesting because of this connection.

Specifically, probability matrices have several special properties:

(1) Any probability matrix has 1 as an eigenvalue.
(2) If all of the entries of a probability matrix are nonzero, then the dimension of the eigenspace for the eigenvalue 1 is 1; i.e. there is precisely one vector v (up to multiplying by a scalar) such that $Pv = v$. We call this vector a **stable vector** for P – you can think of this as a state which is stable under iterating the states according to the matrix P.
(3) All eigenvalues of a probability matrix are ≤ 1 – you will show this on your HW this week!

We will prove some of these properties about such matrices next week.