SQUARE ROOTS OF MATRICES, GRAPHS, AND ADJACENCY MATRICES

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1. Random Question

Question 1.1. *Can you place 4 points in the plane such that any two points are an odd distance apart?*

2. Last Week's HW

Average was about 90/100 – consequently, there wasn't much to really talk about. Most students seemed to be comfortable with the basic concepts; however, there was some confusion in notation that ran rampant through the sets. Specifically, when many students talked about a collection of eigenvectors that spanned a space, they would write the collection of vectors as a single matrix: while I understood what you were talking about and refrained from deducting points, this is incorrect (as a matrix, technically speaking, isn't spanning anything.) In the future / **on the final!**, make sure you don't do this, and write a collection of vectors as, well, a collection of vectors (i.e. $\langle(0,1,2),(1,2,3),(2,2,2)\rangle$, not $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}$.)

3. Square Roots of Matrices

So: when we study numbers, we are often interested in finding solutions to equations like

$$x^n = y \tag{3.1}$$

for given y – i.e. finding n-th roots of numbers. As mathematicians, we are interested in doing something similar for matrices – i.e. finding conditions under which we can find matrices B such that

$$B^n = A \tag{3.2}$$

for some given matrix A. We think of such matrices as n-th roots of A, and we know from class/the online notes posted by Wilson that such roots exist whenever A is a **positive semdefinite** matrix. In case you've forgotten, we repeat the definition of positive semdefinite below:

Definition 3.3. We say that a $n \times n$ matrix A is **positive semdefinite** if for any real n-dimensional vector x,

$$x^T A x \geq 0. \tag{3.4}$$

A nice consequence of being positive semidefinite is that the matrix A is diagonalizable: i.e. that there is an invertible matrix E formed out of A's eigenvectors and a diagonal matrix D made of A's eigenvalues such that

$$A = EDE^{-1}, \tag{3.5}$$

and furthermore that the values in the diagonal matrix are all positive.

Given this, we can easily calculate a n-th root for A by setting

$$B = E\sqrt[n]{D}E^{-1}, \tag{3.6}$$

as

$$B^n = E(\sqrt[n]{D})^n E^{-1} = EDE^{-1} = A \tag{3.7}$$

where the n-th root of D is just $\begin{pmatrix} \sqrt[n]{\lambda_i} & \dots & 0 \\ \vdots & \ddots & \\ 0 & \dots & \sqrt[n]{\lambda_n} \end{pmatrix}$, the coördinate-wise root of D.

So: to illustrate the general method, we work an example below:

Question 3.8. *What is the square root of*

$$A = \begin{pmatrix} 5/2 & -3/2 \\ -3/2 & 5/2 \end{pmatrix}?$$

Proof. So: we begin by first noting that such a matrix is positive definite, as

$$\begin{aligned} x^T \begin{pmatrix} 5/2 & -3/2 \\ -3/2 & 5/2 \end{pmatrix} x = x^T \begin{pmatrix} 5x_1/2 - 3x_2/2 \\ -3x_1/25x_2/2 \end{pmatrix} &= 5x_1^2/2 - 3x_2x_1 + 5x_2^2/2 \\ &= 5/2(x_1^2 + x_2^2 - 6x_2x_1/5) \\ &= 5/2((x_1 - x_2)^2 + 4x_2x_1/5) \geq 0, \end{aligned}$$

because $|(x_1 - x_2)^2| \geq |x_1x_2| \geq 4/5|x_1x_2|$ for all x.

Given this, we know that we can diagonalize A and write it in the form EDE^{-1}, where E is a matrix corresponding to the eignvectors of A and D is the diagonal matrix made out of eigenvalues.

So: it suffices to simply find the eigenvalues/vectors and construct these matrices! To find the eigenvalues, simply note that

$$A - (1)I = \begin{pmatrix} 5/2 - 1 & -3/2 \\ -3/2 & 5/2 - 1 \end{pmatrix} = \begin{pmatrix} 3/2 & -3/2 \\ -3/2 & 3/2 \end{pmatrix}$$

is singular and has null space spanned by $(1, 1)$, and

$$A - (4)I = \begin{pmatrix} 5/2 - 4 & -3/2 \\ -3/2 & 5/2 - 4 \end{pmatrix} = \begin{pmatrix} -3/2 & -3/2 \\ -3/2 & -3/2 \end{pmatrix}$$

is singular and has null space spanned by $(-1, 1)$.

So the eigenvalues are 1 and 4 and have corresponding orthonormal eigenvectors $(1/\sqrt{2}, 1/\sqrt{2})$, $(-1/\sqrt{2}, 1/\sqrt{2})$; so we get that

$$E = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \text{ the 45-degree rotation matrix,}$$

$$E^{-1} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \text{ the -45-degree rotation matrix,}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix},$$

$$\text{and } A = EDE^{-1};$$

so we can write

$$\sqrt{A} = E\sqrt{D}E^{-1}.$$

Double-checking to make sure that this method works gives us

$$\sqrt{A}^2 = (E\sqrt{D}E^{-1})^2 = E(\sqrt{D})^2E^{-1} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -4/\sqrt{2} & 4/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 5/2 & -3/2 \\ -3/2 & 5/2 \end{pmatrix}.$$

So, it works! □

4. Adjacency Matrices

Definition 4.1. So: given any $n \times n$ probability matrix

$$A = \{a_{ij}\},$$

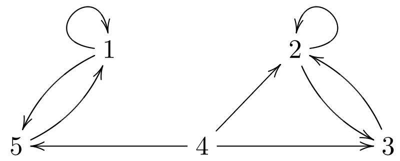
we can form the adjacency graph (V_A, E_A) to A by defining

- the collection of vertices, V, to be some enumerated set of points $\{1, 2 \ldots n\}$, and
- the collection of edges, E, to be the collection of all ordered pairs (n, m) such that a_{mn} is nonzero. (note that this is backwards from what you might normally write – this is because of the column-stochastic thing versus the row-stochastic thing.)

An example graph would be

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix},$$

which would have corresponding graph



So: note the following really useful fact:

Proposition 4.2. *For a matrix A with adjacency graph (V_A, E_A), there is a path of length k from the point m to the point n iff the entry in the n-th row and m-th column of A^k is nonzero. Put another way, if an entry $a_{n,m}$ in the matrix A^k is nonzero, there is a path of length k from m to n.*

Proof. The proof for this is relatively basic, and goes by induction. The base case is trivial, as A^1 corresponds precisely to paths of length 1 in the adjacency graph; so suppose it holds for A^n. Then, simply write

$$A^{n+1} = A^n \cdot A,$$

and note that the n, m-th entry pf the matrix A^{n+1} is nonzero if the dot product of the n-th row and m-th column is nonzero. This holds if and only if there is a k such that the n, k-th entry in A^n is nonzero and the k, m-th entry in A is nonzero: by inductive hypothesis, this means that there is a path of length n from k to n and a path of length 1 from m to k. Composing gives a path of length $n + 1$ from n to m; so we are done! □

This can be used to characterize strongly connected graphs nicely – specifically, a strongly-connected graph must have that for every pair m, n there is a power of k such that the n, m-th entry of A^k is nonzero, as strongly connected graphs must have paths from any node to any other node.