| Math 1b | TA: Padraic Bartlett |
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# Recitation 2: Nonnegative Solutions and Dependence

*Week 2* | *Caltech 2011*

## 1 Random Question

For any $n \in \mathbb{N}$, can you find a matrix $M$ such that

- $M^n = 0$, the all-zeros matrix, but
- $M^k \neq 0$, for any $1 \leq k \leq n$?

(By $M^n$, we simply mean the matrix acquired by multiplying $M$ by itself $n$ times.)

## 2 HW comments

- Section average: 90%.
- As the above indicates, people did pretty well! The only thing I'd mention is that, in general, try to show more of your work! If the total body of your work on a question is just a matrix, it's very difficult to tell *why* your answer is what it is: in particular, if you make a mistake, it becomes impossible to tell the difference between just forgetting a sign early in your calculations (a minor mistake) and flat-out not having any idea how to attack the question at all (a larger problem.) So write more!

## 3 Nonnegative Solutions to Systems of Linear Equations

Last week, we outlined how to find solutions $\mathbf{x}$ to the system

$$A\mathbf{x} = \mathbf{b}.$$

Today, we're going to explore a refinement to the above question: specifically, suppose we want to find solutions to the system $A\mathbf{x} = \mathbf{b}$ where the vector $\mathbf{x}$ is made up of *nonnegative* solutions? Can we do this?

In fact, we can! Specifically, we have the following algorithm:

1. We start with a matrix $M = (A|\mathbf{b})$, that corresponds to our system of linear equations as follows:

$$\begin{bmatrix} a_{11}y_1 & + & \dots & a_{1n}y_n & = & b_1 \\ a_{21}y_1 & + & \dots & a_{2n}y_n & = & b_2 \\ \vdots & & \ddots & \vdots & & \vdots \\ a_{m1}y_1 & + & \dots & a_{mn}y_n & = & b_m \end{bmatrix} \longrightarrow \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ a_{21} & \dots & a_{2n} & b_2 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right) = M$$

2. Now, take this matrix $M$ and via row operations, manipulate it so that the $A$-portion of $M$ is in reduced row-echelon form. Call this reduced matrix $M'$. Check to make sure that this system of equations is consistent (as otherwise, there aren't any solutions at all!)

3. Now, examine the **b**-column in $M'$. There are three possibilities:

   (a) All of the entries in **b** are nonnegative. In this situation, you've found a nonnegative solution, by setting all of the free variables in our system of equations to 0 (as this forces the fixed variables to take their values from the values in **b**.)

   (b) There is an entry $b_k$ in **b** that is negative, but there are no other negative entries in the $k$-th row of $M'$. In this case, because the $k$-th row of our matrix represents the equation

   $$a'_{k1}y_1 + \ldots 1'_{kn}y_n = b_k,$$

   we know that (because all of the $a_{kj}$'s are positive and the entry $b_k$ is negative) one of the $y_j$'s has to be negative for this equation to hold – i.e. any solution to our system must have a negative entry in it. So no nonnegative solution is possible.

   (c) There is an entry $b_k$ in **b** that is negative, and also an entry $a_{kj}$ in the same row as $b_k$ that's negative. If this holds, pivot at $a_{kj}$ and repeat step (3) again. (Sometimes, you will have multiple choices for $a_{kj}$. If this happens, just make sure that pick your pivot entries so that you're not getting into a "loop" – i.e. continually repeating the same set of three pivots again and again and again.)

Proving that this algorithm will always terminate in a finite number of steps is not a terribly tricky thing to do: try it if you have time! (Or ask me; I can prove it in office hours pretty easily.)

Instead, let's run the algorithm twice, so we can see how it works in practice:

**Example.** Does the system of equations represented by the matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 2 & 6 & 4 \\ 1 & 1 & 0 & 0 \end{array}\right)$$

have a nonnegative solution?

**Proof.** So: let's run our algorithm. First, we put the matrix above into reduced row-echelon form, by first pivoting at (1,1):

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 2 & 6 & 4 \\ 0 & -1 & -3 & -2 \end{array}\right),$$

and then at (2,2):

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Now that we're in reduced row-echelon form, we look at the **b**-column. The last column has a negative entry at $b_1$. Thus, we now look at the first row: is there a negative entry in it? As it turns out, there is, at (1,3) – so, let's pivot there!

$$\left(\begin{array}{ccc|c} -1/3 & 0 & 1 & 2/3 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The **b**-column of this matrix is now nonnegative – so there is a basic nonnegative solution! Specifically, if we let the free variable ($x$) be 0, we can then see that the fixed variables $y$ and $z$ are forced to be 0 and 2/3 respectively, and thus that we have $(0, 0, 2/3)$ as a basic nonnegative solution to our system.

**Example.** Does the system of equations represented by the matrix

$$\left(\begin{array}{cccc|c} 3 & 1 & 4 & 1 & 5 \\ 9 & 2 & 6 & 5 & 3 \end{array}\right)$$

have a nonnegative solution?

**Proof.** Again, let's run our algorithm. First, we put the matrix above into reduced row-echelon form, by first pivoting at (1,1):

$$\left(\begin{array}{cccc|c} 1 & 1/3 & 4/3 & 1/3 & 5/3 \\ 0 & -1 & -6 & 2 & -12 \end{array}\right),$$

and then at (2,2):

$$\left(\begin{array}{cccc|c} 1 & 0 & -2/3 & 1 & -7/3 \\ 0 & 1 & 6 & -2 & 12 \end{array}\right).$$

Again, now that we're in reduced row-echelon form, we look at the **b**-column. The last column has a negative entry at $b_1$; so we again look at the first row. Because $(1, 3)$ has a negative value in it, we pivot there:

$$\left(\begin{array}{cccc|c} -3/2 & 0 & 1 & -3/2 & 7/2 \\ 9 & 1 & 0 & 7 & -9 \end{array}\right).$$

Again, there is a negative entry in the **b**-column of the above matrix – but in the corresponding row of our matrix, there aren't any other negative entries! Therefore, we know that there are no nonnegative solutions to our system.

## 4 Dependence

In linear algebra, there are a number of concepts of "dependence." We list two common ones here:

**Definition. Linear dependence**: A collection $v_1 \dots v_k$ of vectors is called **linearly dependent** iff there are $k$ constants $a_1 \dots a_k$, not all identically 0, such that

$$\sum_{i=1}^{k} a_i v_i = 0.$$

**Definition. Affine dependence**: A collection $v_1 \dots v_k$ of vectors is called **affinely dependent** iff there are $k$ constants $a_1 \dots a_k$, not all identically 0, such that

$$\sum_{i=1}^{k} a_i v_i = 0,$$

and

$$\sum_{i=1}^{k} a_i = 0.$$

Along with those two definitions, we have a pair of theorems to tell us when a collection of vectors is either linearly or affinely dependent:

**Theorem 1** *Suppose that you have a collection of vectors $v_1 \dots v_k$, and you use them to create the matrix*

$$A = \begin{pmatrix} \dots & v_1 & \dots \\ \dots & v_2 & \dots \\ & \vdots & \\ \dots & v_k & \dots \end{pmatrix}$$

*by taking the vectors $v_1 \dots v_k$ as the rows of $A$. Then, this collection $v_1 \dots v_k$ of vectors is linearly dependent if and only if the reduced echelon form of the matrix $A$ has a zero row in it.*

The proof of this theorem was provided in class (I believe; if not, prove it!) Similarly, on HW#2, you're asked to prove a theorem that tells you when a collection of vectors is affinely dependent:

**Theorem 2** *A collection of vectors $v_1 \dots v_k$ is affinely dependent iff the collection of vectors $v_2 - v_1, v_3 - v_1, \dots v_n - v_1$ is linearly dependent.*