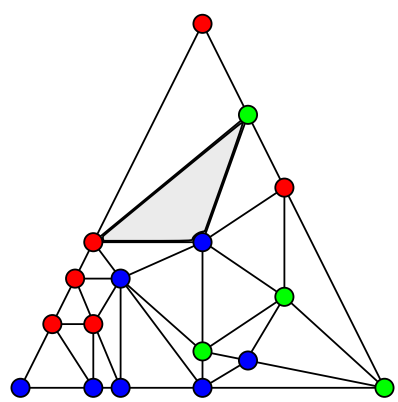
Recitation 8: Graphs and Adjacency Matrices

1 Random Question

Suppose you take a large triangle ΔXYZ, and divide it up with straight line segments into a bunch of smaller triangles. Now, suppose you color the vertices of this triangle as follows:

- Vertex X is colored red,
- Vertex Y is colored blue,
- Vertex Z is colored green,
- All of the vertices on the line $\overline{XY}$ are either red or blue,
- All of the vertices on the line $\overline{YZ}$ are either blue or green, and finally
- All of the vertices on the line $\overline{ZX}$ are either green or red.

Show that you must have colored one of the small triangles with all three colors: i.e. that there is a small triangle with one red vertex, one green vertex, and one blue vertex.



2 Homework comments

- Section average: 90%. This was about 5% above the class average; nice work!
- People did pretty well, all in all! There were two reoccuring errors that were somewhat concerning:
  - Several people wrote that $(A^TA)^T = AA^T$. This is not true! I've said it like fifteen times by now, but: for square matrices, $(AB)^T = B^TA^T$. So, specifically, $(A^TA)^T = (A)^T(A^T)^T = A^TA$; i.e. it is its own transpose! It's an easy mistake to make, but do be careful.

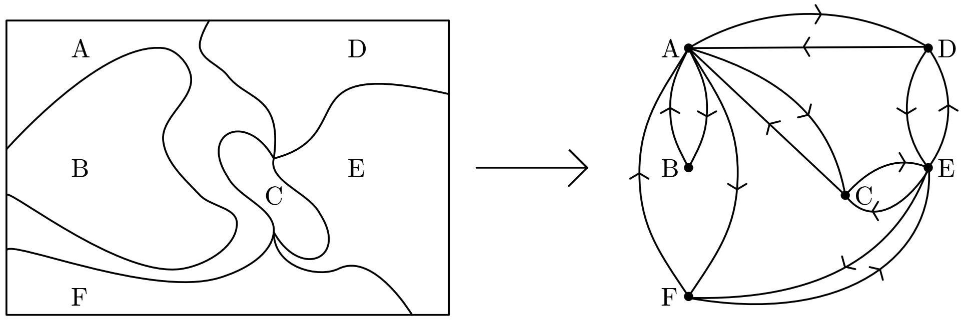
– As we said at the end of last recitation: not all matrices are diagonalizable! For example, the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable; in other words, there is no invertible matrix E and diagonal matrix D such that $A = EDE^{-1}$. (This is because the matrix A has only one eigenvector, up to scalar multiples.)

3 Graphs and Matrices: Definitions and Examples

Definition. A **directed graph** $G = (V, E)$ consists of the following:

- A set V, which we call the set of *vertices* for G, and
- A set $E \subset V^2$, made of ordered pairs of vertices, which we call the set of *edges* for G.

Example. In the map below, take the set of the six labeled countries $\{A, B, C, D, E, F\}$ as our vertices, and connect two of these vertices with an edge if and only if their corresponding countries share a border of any positive length. This then defines a graph, which we draw to the right of our map:



Definition. Given a graph $G = (V, E)$ on the vertex set $V = \{1, 2, \ldots n\}$, we can define the **adjacency matrix** for G as the following $n \times n$ matrix:

$$A_G := \left\{ a_{ij} \;\middle|\; \begin{array}{ll} a_{ij} = 1 & \text{if the edge } (i,j) \text{ is in } E; \\ a_{ij} = 0 & \text{otherwise.} \end{array} \right\}$$

Example. If we let G be the map-graph we drew above in our earlier example, then A_G is the following matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Remark. Conversely, given a $n \times n$ matrix A, we can create a graph G_A associated to said matrix by letting

- $V = \{1, \dots n\}$, and
- $E = \{(i,j) : a_{ij} \neq 0\}$.

It bears mentioning that this process effectively "undoes" our adjacency-matrix operation above: i.e. if we start with a graph G, take its adjacency matrix A_G, and then turn this matrix back into a graph G_{A_G}, we have the same graph. Similarly, if we start with a matrix made entirely out of zeroes and ones, turn it into a graph, and then back into matrix, we will have changed nothing.

As the definitions above illustrate, we have ways of converting graphs into matrices, and vice-versa. A good question to ask now is the following: why do we care? In other words, what do we gain by being able to turn graphs into matrices, or vice-versa?

We answer this question with our next section, which shows us how the adjacency matrix for a graph can shed a lot of light on the graph itself:

4 Graphs and Matrices: A Theorem and its Use

Theorem 1 *Suppose that $G = (V, E)$ is a graph with $V = \{1, \dots n\}$, A_G is its corresponding $n \times n$ adjacency matrix, and i, j are a pair of vertices in G.*

Then there is a path[1] of length m from i to j if and only if the (i, j)-th entry of $(A_G)^m$ is nonzero.

Proof. To prove this claim, we first make the following inductive observation:

Claim 2 *The (i, j)-th entry of $(A_G)^m$ is the following sum:*

$$\sum_{c_1, c_2, \dots c_{m-1}=1}^{n} a_{i,c_1} \cdot a_{c_1,c_2} \cdot \ldots \cdot a_{c_{m-1},j}$$

Proof. We prove this claim by induction on m. For $m = 1$, this is trivially clear: $a_{ij} = a_{ij}$.

So: we assume that this holds for m, and seek to prove our claim for $m+1$: i.e we seek to show that the $(i, j$-th entry of $(A_G)^{m+1}$ is

$$\sum_{c_1, c_2, \dots c_m=1}^{n} a_{i,c_1} \cdot a_{c_1,c_2} \cdot \ldots \cdot a_{c_m,j}.$$

So: to do this, write $(A_G)^{m+1} = A_G \cdot (A_G)^m$. Then we can write the (i, j)-th entry of $(A_G)^{m+1}$ as the i-th row of A_G times the j-th column of $(A_G)^m$. We know that the i-th row

[1]For a graph G, a **path** from i to j of length m is a sequence of vertices $(i, c_1, c_2 \dots c_{m-1})$ and a corresponding sequence of edges $((i, c_1), (c_1, c_2), \dots (c_{m-1}, j))$. You can think of this as describing a way to "walk" from i to j while using precisely m edges. Note that your path is allowed to cross over itself; we are not asking that any of these edges or vertices are distinct, merely that they *exist*.

of A_G is just $(a_{i,1}, \ldots a_{i,n})$, and by our inductive hypothesis we know that the j-th column of $(A_G)^m$ is

$$\begin{pmatrix} \sum_{c_1,c_2,\ldots c_{m-1}=1}^n a_{1,c_1} \cdot \ldots \cdot a_{c_m,j} \\ \sum_{c_1,c_2,\ldots c_{m-1}=1}^n a_{2,c_1} \cdot \ldots \cdot a_{c_m,j} \\ \vdots \\ \sum_{c_1,c_2,\ldots c_{m-1}=1}^n a_{n,c_1} \cdot \ldots \cdot a_{c_m,j} \end{pmatrix}$$

So, if we take the product of this row and column, we have that the (i,j)-th entry of $(A_G)^{m+1}$ is

$$\begin{aligned} &(a_{i,1}, \ldots a_{i,n}) \cdot \begin{pmatrix} \sum_{c_1,c_2,\ldots c_{m-1}=1}^n a_{1,c_1} \cdot \ldots \cdot a_{c_m,j} \\ \sum_{c_1,c_2,\ldots c_{m-1}=1}^n a_{2,c_1} \cdot \ldots \cdot a_{c_m,j} \\ \vdots \\ \sum_{c_1,c_2,\ldots c_{m-1}=1}^n a_{n,c_1} \cdot \ldots \cdot a_{c_m,j} \end{pmatrix} \\ &= \left(a_{i,1} \cdot \sum_{c_1,c_2,\ldots c_{m-1}=1}^n a_{1,c_1} \cdot \ldots \cdot a_{c_m,j} \right) + \ldots + \left(a_{i,n} \cdot \sum_{c_1,c_2,\ldots c_{m-1}=1}^n a_{n,c_1} \cdot \ldots \cdot a_{c_m,j} \right) \\ &= \left(\sum_{c_1,c_2,\ldots c_{m-1}=1}^n a_{i,1} \cdot a_{1,c_1} \cdot \ldots \cdot a_{c_m,j} \right) + \ldots + \left(\sum_{c_1,c_2,\ldots c_{m-1}=1}^n a_{i,n} \cdot a_{n,c_1} \cdot \ldots \cdot a_{c_m,j} \right) \\ &= \sum_{k,c_1,c_2,\ldots c_{m-1}=1}^n a_{i,k} \cdot a_{k,c_1} \cdot \ldots \cdot a_{c_m,j}. \end{aligned}$$

If we relabel the constants $k, c_1, \ldots c_{m-1}$ appropriately, this is precisely the sum we wanted; therefore, we've proven our inductive claim.

So: we've just proven that the (i,j)-th entry of $(A_G)^m$ is

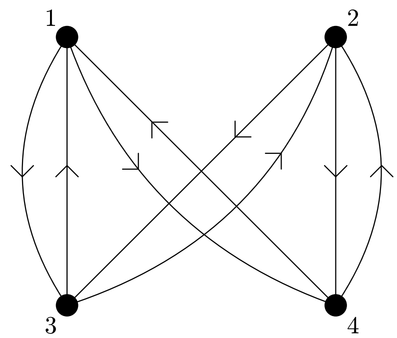
$$\sum_{c_1,c_2,\ldots c_{m-1}=1}^n a_{i,c_1} \cdot a_{c_1,c_2} \cdot \ldots \cdot a_{c_{m-1},j}.$$

What does this mean? Well, suppose that there was a path of length m from i to j. Then there must be vertices $c_1 \ldots c_{m-1}$ such that the edges $(i, c_1), \ldots (c_{m-1}`, j)$ all exist; in other words, all of the values $a_{i,c_1}, \ldots a_{c_{m-1},j}$ must be 1. So their product is also 1; therefore, at least one of the elements in the sum above is nonzero! Because all of the a_{kl}'s are nonnegative, this means that the entire sum above is positive! Thus, we've just shown that if there *is* a path from i to j of length m, then the (i,j)-th entry is nonzero.

Conversely, suppose that the (i,j)-th entry is nonzero. The only way this can happen is if one of the terms in the sum above is nonzero: i.e. if there are vertices $c_1, \ldots c_{m-1}$ such that $a_{i,c_1} \cdot a_{c_1,c_2} \cdot \ldots \cdot a_{c_{m-1},j} \neq 0$. But the only way for this to happen is if all of their individual values are 1; i.e. if all of the edges $(i, c_1), \ldots (c_{m-1}`, j)$ exist! As this constitutes a path of length m from i to j, we've just proven the other direction of our claim: that if the (i,j)-th entry of $(A_G)^m$ is nonzero, then there is a path of length m. So we're done!

If you don't care about the proof, the following question should be more interesting:

Question 3 *Is there a path of length* 243 *from* 1 *to* 2 *in the following graph?*



Solution. Call the graph above G, and notice that A_G has the following form:

$$A_G = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

By our theorem above, we know that a path of length 243 from 1 to 2 can exist iff the (1,2)-entry of $(A_G)^{243}$ is nonzero. Thus, we've reduced our path-finding question to one of raising matrices to large powers – something we showed last week is trivial for diagonalizable matrices!

So, let's see if this matrix is diagonalizable. To do this, we first find its characteristic polynomial:

$$\begin{aligned} p_{A_G}(\lambda) &= \det(\lambda I - A_G) \\ &= \det \begin{pmatrix} \lambda & 0 & -1 & -1 \\ 0 & \lambda & -1 & -1 \\ -1 & -1 & \lambda & 0 \\ -1 & -1 & 0 & \lambda \end{pmatrix} \\ &= \lambda \cdot \det \begin{pmatrix} \lambda & -1 & -1 \\ -1 & \lambda & 0 \\ -1 & 0 & \lambda \end{pmatrix} - \det \begin{pmatrix} 0 & \lambda & -1 \\ -1 & -1 & 0 \\ -1 & -1 & \lambda \end{pmatrix} + \det \begin{pmatrix} 0 & \lambda & -1 \\ -1 & -1 & \lambda \\ -1 & -1 & 0 \end{pmatrix} \\ &= \lambda(\lambda^3 - 2\lambda) - (\lambda^2) - (\lambda^2) \\ &= \lambda^2(\lambda - 2)(\lambda + 2). \end{aligned}$$

Therefore, our matrix has eigenvalues $0, 2, -2$. To find their multiplicity, we investigate the appropriate nullspaces, which we find by just being clever and noticing certain combinations of the columns which go to zero:

$$
\begin{aligned}
E_0 &= \text{nullspace}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} && = \text{span}\{(1,-1,0,0),(0,0,1,-1)\} \\
E_2 &= \text{nullspace}\begin{pmatrix} -2 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & -2 \end{pmatrix} && = \text{span}\{(1,1,1,1)\} \\
E_{-2} &= \text{nullspace}\begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix} && = \text{span}\{(-1,-1,1,1)\}
\end{aligned}
$$

Thus, we have four linearly eigenvectors! This allows us to write A_G as the following product of matrices:

$$
\begin{aligned}
A_G &= \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 \end{pmatrix}^{-1} \\
\Rightarrow \quad (A_G)^{243} &= \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2^{243} & 0 & 0 & 0 \\ 0 & (-2)^{243} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 \end{pmatrix}^{-1}
\end{aligned}
$$

So, you *could* just multiply this all out and see what you get; however, 2^{243} is really really *really* big, and can kind of crash a lot of programs! Thus, what you might want to try instead is noticing the following trick: the diagonal matrix in the middle? That's just $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ times the constant 2^{242}: therefore, because scalar multiplication commutes with matrices, we can write

$$\begin{aligned}
(A_G)^{243} &= \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2^{243} & 0 & 0 & 0 \\ 0 & (-2)^{243} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 \end{pmatrix}^{-1} \\
&= 2^{242} \cdot \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & -1 \end{pmatrix}^{-1} \\
&= 2^{242} \cdot A_G.
\end{aligned}$$

Therefore, we know that the (1,2)-entry of $(A_G)^{243}$ is nonzero iff the (1,2)-entry in A_G itself is nonzero! By inspection, it is 0; therefore, we know that there can be no path of length 243 from 1 to 2.

It bears noting that there was nothing special about 243 in the above proof, other than the fact that it was odd, and thus allowed us to factor $\begin{pmatrix} 2^{243} & 0 & 0 & 0 \\ 0 & (-2)^{243} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ into $2^{242} \cdot \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Had our number been even, of course, this would not have worked: raising the (-2) in our matrix to an even power would have made it positive, and we would have had no way to do the factoring trick we just did. (This is because there are, in fact, even-length paths of any positive length from 1 to 2! Looking at the graph for a while should convince you of this fact, if you don't want to do the matrix multiplication...)