FINAL REVIEW! - SELECTED EXERCISES

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1. Exam Properties

So: the final will cover chapters 5-8 with the exceptions of sections 6.4,7.7,8.5,8.6 – i.e. the material covered in the fifth through eighth homeworks. Basically, what you need to know is

- Chapter 5 – how to do basic integration; Fubini's theorem.
- Chapter 6 – Change of Variables formula – the general form for 2 and 3 dimensions, as well as the explicit transformations for polar, cylindrical and spherical coördinates; also, how to use integrals to calculate average values and centers of mass.
- Chapter 7 – different ways of taking integrals; i.e. how to integrate functions and vector fields over curves and surfaces.
- Chapter 8 – Green's theorem, the divergence theorem, Stokes's theorem, and Gauss's theorem.

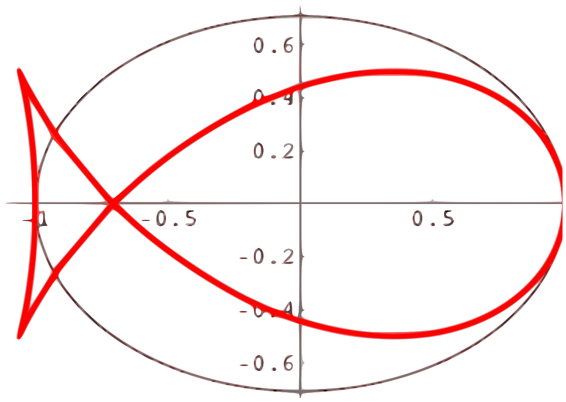
Explicit lists of definitions/theorems and their properties can be found in the earlier notes here.

So: we work a series of examples below, to illustrate the theory we've learned so far.

2. Area of a Fish

Question 2.1. *Find the area bounded by the "fish curve" parametrized by*

$$c(t) = (\cos(t) - \frac{\sin^2(t)}{\sqrt{2}}, \cos(t)\sin(t)).$$



Proof. So: recall the formula for area that's given by Green's theorem: i.e. for D a region bounded by the simple closed curve c^+ oriented positively (i.e. so that the region D is on the LHS of the curve), we have

$$A(S) = \frac{1}{2}\int_{c^+} xdy - ydx.$$

So, we can't apply this directly to c, as this curve is not a simple closed curve! Indeed, $c(\pi/2) = c(3\pi/2)$. However, what we can do is use this formula to find the area of the "head" and the "tail," and simply sum these two areas together.

So: the head is parametrized positively by the curve c on the interval $[-\pi/2, \pi/2]$, and the tail is parametrized negatively by the curve c on the interval $[\pi/2, 3\pi/2]$; you can see this by drawing the curve c from 0 to 2π and drawing little arrows to show you which direction you're going.

As a result, we have that the area of the head is just

$$\frac{1}{2}\int_{-\pi/2}^{\pi/2}(c_1(t)c_2'(t) - c_2(t)c_1'(t))dt$$

and of the tail is

$$-\frac{1}{2}\int_{\pi/2}^{3\pi/2}(c_1(t)c_2'(t) - c_2(t)c_1'(t))dt.$$

(where the minus sign comes from the reversed orientation of the tail.)

So: we calculate!

$$\begin{aligned}
&\frac{1}{2}\int_a^b (c_1(t)c_2'(t) - c_2(t)c_1'(t))dt\\
=&\frac{1}{2}\int_a^b \left(\cos(t) - \frac{\sin^2(t)}{\sqrt{2}}\right)(\cos^2(t) - \sin^2(t)) - (\cos(t)\sin(t))\left(-\sin(t) - \frac{2\sin(t)\cos(t)}{\sqrt{2}}\right)dt\\
=&\frac{1}{2}\int_a^b \cos^3(t) + \frac{\sin^4(t)}{\sqrt{2}} - \frac{\sin^2\cos^2(t)(t)}{\sqrt{2}} + \frac{2\sin^2(t)\cos^2(t)}{\sqrt{2}}dt\\
=&\frac{1}{2}\int_a^b \cos^3(t) + \frac{\sin^2(t)}{\sqrt{2}}dt\\
=&\frac{1}{2}\int_a^b \frac{3\cos(t) - \cos(3t)}{4} + \frac{1-\cos(2t)}{2\sqrt{2}}dt\\
=&\frac{1}{8}\left(3\sin(t) + \frac{\sin(3t)}{3} + t\sqrt{2} - \frac{\sin(2t)}{\sqrt{2}}\right)\Big|_a^b.
\end{aligned}$$

Evaluating this at $a = -\pi/2, b = \pi/2$ gives that the area of the head is $2/3 + \pi\sqrt{2}/8$; evaluating at $a = \pi/2, b = 3\pi/2$ yields that the area of the tail is $-2/3 + \pi\sqrt{2}/8$; combining yields that the entire area is $\pi\sqrt{2}/4$. □

3. Vector fields over a Lissajous curve

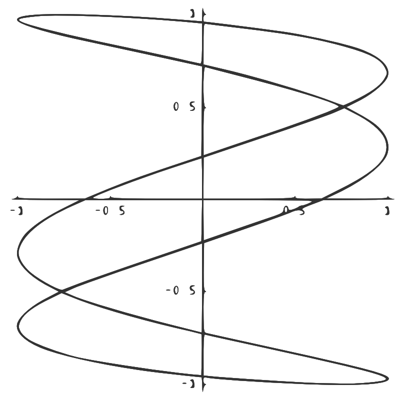
Question 3.1. *For F the vector field defined by*

$$F(x, y, z) = (x^2, y^2, z^2)$$

and $c(t)$ the Lissajous curve parametrized by

$$c(t) = (\sin(3t + \pi/4), \sin(t)),$$

find $\int_c F ds$.



Proof. So, if we merely directly calculate, we have that

$$\begin{aligned}\int_C F ds &= \int_0^{2\pi} (\sin^2(3t+\pi/4), \sin^2(t), 0) \cdot (3\cos(3t+\pi/4), \cos(t), 0) \\ &= \int_0^{2\pi} 3\cos(3t+\pi/4)\sin^2(3t+\pi/4) + \cos(t)\sin^2(t) dt \\ &= \int_0^{2\pi} 3\cos(3t+\pi/4)\sin^2(3t+\pi/4) dt + \int_0^{2\pi} \cos(t)\sin^2(t) dt \\ &= \int_{1/\sqrt{2}}^{1/\sqrt{2}} u^2 du + \int_0^0 v^2 dv = 0,\end{aligned}$$

where the substitutions in the last step were $u = \sin^(3t + \pi/4)$ and $v = \sin(t)$.

Conversely, you could just notive that F is given by the gradient of the function $f(x, y, z) = \frac{x^3+y^3+z^3}{3}$, and thus that

$$\int_{c'} F ds = \int\int \nabla \times (\nabla f) ds = \int\int 0 = 0$$

for any simple closed curve c' (as the curl of a gradient is always 0). Breaking up our Lissajous curve into three simple closed curves then gives that the integral of F over c is 0, as expected. □

4. Integral tricks - I

Question 4.1. *Calculate*

$$\int\int_S x^2 + y^2 z - z^3/3 dxdydz,$$

where S is the unit sphere.

Proof. So, we can directly calculate this with the spherical coördinate transformation $(\theta, \phi) \mapsto (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$

$$\begin{aligned}
&\int\int_S x^2 + y^2 z - z^3/3 dxdydz = \\
=&\int_0^{2\pi}\int_0^{\pi} \cos^2(\theta)\sin^3(\phi)\cos^2(\theta) + \sin^2(\theta)\sin^3(\phi)\cos(\phi) - \frac{\cos^3(\phi)\sin(\phi)}{2} d\phi d\theta \\
=&\int_0^{2\pi}\left(\int_0^{\pi} \cos^2(\theta)\left(\frac{3\sin(t)-\sin(3t)}{4}\right)d\phi + \int_0^{\pi}\sin^2(\theta)\sin^3(\phi)\cos(\phi)d\phi - \int_0^{\pi}\frac{\cos^3(\phi)\sin(\phi)}{2}d\phi\right)d\theta \\
=&\int_0^{2\pi}\cos^2(\theta)\cdot\frac{4}{3} + 0 + 0 \\
=&\frac{4}{3}\pi.
\end{aligned}$$

Alternately, you can notice that

$$\int\int_S x^2 + y^2 z - z^3/3 dxdydz = \int\int_S (x, yz, -\frac{z^2}{2}) \cdot (x, y, z) dxdydz;$$

because the unit normal vector on the sphere is $n(x, y, z) = (x, y, z)$, we know that this is actually

$$\int\int_S (x, yz, -\frac{z^2}{2}) \cdot n dxdydz$$

and thus that we can apply Gauss's theorem to get

$$\int\int_S (x, yz, -\frac{z^2}{2}) \cdot n dxdydz = \int\int\int_B 1 + z - z dxdydz = \int\int\int_B ds = \frac{4}{3}\pi,$$

the volume of the unit ball. □

5. Integral tricks – II

Question 5.1. *Calculate*

$$\int\int_S (2z, 0, 2y) \cdot dS$$

where S is the unit sphere.

Proof. So, we can, again, directly calculate this with the spherical coördinate transformation $(\theta, \phi) \mapsto (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi))$

$$\int\int_S (2z, 0, 2y) dxdydz =$$

$$= \int_0^{2\pi} \int_0^{\pi} (2\cos(\phi), 0, 2\sin(\phi)\sin(\theta)) \cdot (-\sin^2(\phi)\cos(\theta), \sin^2(\phi)\cos(\theta), -\sin(\phi)\cos(\theta)) d\phi d\theta$$

$$= \int_0^{2\pi} \left(\int_0^{\pi} -2\cos(\theta) \cdot \cos(\phi)\sin^2(\phi) d\phi - \int_0^{\pi} 2\sin^2(\phi)\sin(\theta)\cos(\theta) d\phi \right) d\theta$$

$$= \int_0^{2\pi} \left(\int_0^{0} -2\cos(\theta) \cdot u^2 du - \int_0^{\pi} \sin^2(\phi)\sin(2\theta) d\phi \right) d\theta$$

$$= -\int_0^{2\pi} \int_0^{\pi} \sin^2(\phi)\sin(2\theta) d\phi d\theta$$

$$= -\int_0^{\pi} \int_0^{2\pi} \sin^2(\phi)\sin(2\theta) d\theta d\phi$$

$$= 0,$$

by using various trig identities, the substitution $u = \sin(\phi)$, and the fact that $\sin(2\theta)$ has integral 0 over $[0, 2\pi]$. Alternately, you can notice that

$$\int\int_S (2z, 0, 2y) dxdydz = \int\int_S \nabla \times (y^2, z^2, 0) dxdydz;$$

applying Gauss's theorem then yields

$$\int\int_S \nabla \times (y^2, z^2, 0) dxdydz = \int\int\int_B div(\nabla \times (y^2, z^2, 0)) dxdydz = 0$$

because the divergence of a curl is always 0.

Finally, you could instead just use use Stokes's theorem, which says also that

$$\int\int_S \nabla \times (y^2, z^2, 0) dxdydz = \int_{\partial S} (y^2, z^2, 0) dxdydz = 0$$

because the unit sphere has no boundary. □