MA1C, WEEK 5: REVIEW! (MIDTERM)

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These notes, like all notes, can be found on my website.

1. Last Week's HW

Average: was around 87%; so, as a result, there isn't too much to say here.

2. Random Question

Question 2.1. *So: suppose that you've been teleported back to Rome, and you've AGAIN found yourself in a gladiatorial arena. (The life of a mathematician is hard.) This arena is in the shape of a perfect circle; at each point with rational argument (i.e .angle) of the circle, there is a lion, which is free to run along the boundary of the triangle but cannot escape the boundary of the triangle due to a complicated system of chains. Suppose that the lions here are all point-lions and are* ***not*** *tigers; suppose further that both you and the lions move at 10m/s, can pivot and change direction instantly, are arbitrarily brilliant, and know no fear. Can you escape from the circle?*

3. Midterm review topics

3.1. **Level curves.** Be aware of what they are: i.e. for a function $f : \mathbb{R}^n \to \mathbb{R}$ and a constant $c \in \mathbb{R}$, a level curve for $f = c$ is simply the graph of all of the points in $\mathbb{R}^n$ such that f evaluated at these points is c. An example is worked in the notes for week 1, and several are done in your text; because everyone seems to know how to do this, we omit an example here.

3.2. **Limits, in the multidimensional setting.** Know how to compute limits, via either $\epsilon - \delta$ arguments or through showing that function is continuous; also, know how to show that a function doesn't have a limit at a point. We work two examples below:

Example 3.1. We claim that

$$f(x, y, z) = \frac{x2 + y^2 + z^2}{|x| + |y| + |z|}$$

has limit 0 at 0.

Proof. So: note that because $|x| = \sqrt{x^2}$, and because square root is concave, that we have

$$f(x, y, z) = \frac{x2 + y^2 + z^2}{|x| + |y| + |z|} \leq \frac{x2 + y^2 + z^2}{\sqrt{x2 + y^2 + z^2}} = \sqrt{x2 + y^2 + z^2}$$

for all points not equal to $(0, 0, 0)$. But $\sqrt{x2 + y^2 + z^2}$ is a continuous function, and it goes to 0 as (x, y, z) goes to 0; so, because f is a strictly postive function, it's bounded at all times between 0 and a function which goes to 0 as it goes to $(0, 0, 0)$. So, by the squeeze theorem, we have that $\lim_{(x,y,z)\to 0} f(x, y, z)$ is 0. □

Example 3.2. We claim that the function

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

has no limit at 0.

Proof. So: along the path given by the line $y = 0$, this function is

$$f(x, 0) = \frac{x \cdot 0}{x^2 + 0^2} = 0$$

and thus the limit as (x, y) goes to zero along this path is 0.

Conversely, along the path given by the line $y = x$, this function is

$$f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

and thus the limit as (x, y) goes to zero along this path is $1/2$. Because these values are different, f cannot have a well-defined limit as (x, y) goes to 0, because if f has a limit a at 0, it must approach a along any path that goes to 0. □

3.3. **Partial derivatives.** Know how they're defined: i.e. for a function $f : \mathbb{R}^n \to \mathbb{R}$, the partial derivative of f with respect to its i^{th} coördinate is

$$\frac{\partial(f)}{\partial x_i}(a_1 \dots a_n) := \lim_{h\to 0} \frac{f(a_1, \dots a_i + h, \dots a_n) - f(a_1 \dots a_n)}{h}.$$

We work an example below:

Example 3.3. Show that

$$f(x, y) = \begin{cases} \frac{x^3y^2}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & \text{otherwise.} \end{cases}$$

has $\frac{\partial f}{\partial x}$ equal to 0 at 0.

Proof. So: by definition, the partial derivative of f at 0 is

$$\lim_{(h)\to 0} \frac{\frac{h^3 \cdot 0^2}{h^2+0^2} - 0}{h} = \lim_{(h)\to 0} \frac{0}{h} = 0.$$

So the partial derivative of this function at 0 exists and is 0. □

3.4. **Total derivatives.** So: the total derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is the $m \times n$ matrix defined by

$$Df(a) = \left[\frac{\partial f_i}{\partial x_j}(a) \right].$$

This definition allows us to define whether a function f is differentiable at a point x_0: we say that this holds whenever

$$\lim_{x \ to 0} \frac{||f(x) - f(a) - Df(a) \cdot (x - a)||}{||x - a||} = 0.$$

A useful fact that we often need is that if a function is C^1, it is differentiable – this saves us the trouble of calculating things like the limit above. The converse, however, is not true: see your text or the notes from week 2 for an example of a differentiable function which is not C^1.

3.5. **Chain rule.** So: the chain rule says that for $f : \mathbb{R}^m \to \mathbb{R}^p, g : \mathbb{R}^n \to \mathbb{R}^m$, g differentiable at x_0, f differentiable at $g(x_0)$, we have that

$$D(f \circ g)(x_0) = (Df)(g(x_0)) \cdot (Dg)(x_0).$$

(the key points being that the domain of f is the range of g, and that both functions are differentiable where it's needed.)

We work an example of how to use the chain rule below:

Example 3.4. Let $f(x, y, z) = xyz$ and $g(t) = (1, t, \sin(t))$. Find $D(f \circ g)$ using the chain rule.

Proof. So: we have that both functions are C^∞, as f is a polynomial and g is component-wise a series of C^∞ functions from $\mathbb{R} \to \mathbb{R}$: as well, the domain of f is $\mathbb{R}^3$, which coincides with the range of g. So we can indeed apply the chain rule, and we then get

$$\begin{aligned} D(f \circ g)(t) &= (Df)(g(t)) \cdot (Dg)(t) \\ &= \begin{pmatrix} yz|_{g(t)} & xz|_{g(t)} & xy|_{g(t)} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ \cos(t) \end{pmatrix} \\ &= \begin{pmatrix} t\sin(t) & \sin(t) & t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ \cos(t) \end{pmatrix} \\ &= \sin(t) + t\cos(t). \end{aligned}$$

So this is the derivative of $f \circ g$. □

3.6. **Higher-order partial derivatives.** So: we defined the higher-order partial derivatifes of a function f recursively by

$$\frac{\partial^m f}{\partial x_{i_1} \dots \partial x_{i_n}} = \frac{\partial}{\partial x_{i_1}} \left(\frac{\partial}{\partial x_{i_2}} \left(\dots \frac{\partial f}{\partial x_{i_n}} \right) \right),$$

where we calculate each individual partial derivative in the normal fashion. Simply know how to do this: also, know that if the function f is C^n, then if we're computing a n-th partial derivative, we can do this in any order we like: i.e. it doesn't matter in which order you differentiate.

3.7. **Extrema.** So: for $f : U \subset \mathbb{R}^n \to \mathbb{R}$, where U is an open set and x_0 is a point in U, we say that x_0 is a **critical point** of f if either $(Df)(x_0) = 0$, (where by $(Df)(x_0) = 0$ we mean that every entry of the $1 \times n$ matrix $(Df)(x_0)$ is 0,) or f doesn't have a defined derivative at this point.

If f is C^2, we can say more: i.e.

- f has a local minimum at x_0 iff x_0 is a critical point and the matrix $\left[\frac{\partial^2 f}{\partial x_i x_j}(x_0)\right]$ has only positive eigenvalues (i.e. $Hf(x_0)$ is positive-definite.)
- f has a local maximum at x_0 iff x_0 is a critical point and the matrix $\left[\frac{\partial^2 f}{\partial x_i x_j}(x_0)\right]$ has only negative eigenvalues (i.e. $Hf(x_0)$ is negative-definite.)
- f has a saddle point at x_0 iff x_0 is a critical point and the matrix $\left[\frac{\partial^2 f}{\partial x_i x_j}(x_0)\right]$ has a positive eigenvalue and a negative eigenvalue.

We work an example below:

Example 3.5. Show that the functions $f_1(x, y) = x^2+y^2$, $f_2(x, y) = x^2-y^2$, $f_3(x, y) = -x^2 - y^2$ have a local minimum, saddle point, and local maximum at $(0, 0)$, respectively.

Proof. So: note that the matrix of second partial derivatives of

- f_1 at (0,0) is $M_1 := \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$,
- f_2 at (0,0) is $M_2 := \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, and
- f_3 at (0,0) is $M_3 := \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$.

As a result, we have that

- all of the eigenvalues of M_1 are positive, so f has a local minimum at (0,0),
- one eigenvalue of M_2 is positive and one is negative, so f has a local minimum at (0,0), and
- all of the eigenvalues of M_3 are negative, so f has a local maximum at (0,0).

□

3.8. **Lagrange Multipliers.** So: the method of Lagrange multipliers is outlined below. Suppose that

- $f, g : U \subseteq \mathbb{R}^n \to \mathbb{R}$ are C^1 functions,
- S is the level set $g(x) = c$, for some constant c, and
- x_0 is a point in S.

Then whenever $f|_S$, the function f restricted to the set S, has a critical point at x_0, there is some $\lambda \in \mathbb{R}$ such that

$$\nabla f(x_0) = \lambda \nabla f(x_0).$$

So: using this method, we can classify all of the critical points of a function f (and thus all potential local minima and maxima) on any closed set U with boundary given by the level curve of some C^1 function g – we can do this by finding all of the critical points on the interior of U by our normal method of classifying extrema (looking for points where $Df = 0$,) and using the method of Lagrange multipliers on the boundary.

We furthermore know that if U is closed and bounded, and f is continuous, that f must attain an absolute maximum and a minimum on U (by some theorem in your text;) so, if we want to find the absolute maximum and minimum of a function on a set, we can simply use the methods above.

We work an example below:

Example 3.6. For $f(x, y) = \frac{x^2 y}{2}$, find all of the critical points of f on the unit disk $\mathbb{D}$, state whether f has a absolute max/min on $\mathbb{D}$, and (if so) find it.

Proof. So: first note that f is continuous on $\mathbb{D}$, and $\mathbb{D}$ is bounded and closed; so f indeed has an absolute maximum and minimum on this set, and furthermore that it is some critical point of f.

So, we break $\mathbb{D}$ into two pieces: the open set of all points (x, y) with $x^2 + y^2 < 1$, and the unit circle $x^2 + y^2 = 1$.

On the first set, we know that the only critical point are those where $Df = (xy, x^2)$ is 0; i.e. all of the points of the form $(0, y)$.

On the second set, we know that the only critical points are those where we can find a λ such that

$$\nabla f = (xy, x^2) = \lambda \cdot \nabla g = (2x, 2y);$$

i.e. (via algebra) the four points $(0, \pm 1)$, $\left(\pm\sqrt{\frac{-1+\sqrt{5}}{2}}, \frac{-1+\sqrt{5}}{2}\right)$, Evaluating gives us that $f(x, y) = 0$ on all points with $x = 0$; for the two critical points $\left(\pm\sqrt{\frac{-1+\sqrt{5}}{2}}, \frac{-1+\sqrt{5}}{2}\right)$, we have that $f(\left(\pm\sqrt{\frac{-1+\sqrt{5}}{2}}, \frac{-1+\sqrt{5}}{2}\right) = \pm\left(\frac{-1+\sqrt{5}}{2}\right)^{3/2}$. As a result, we have that the absolute maximum of f on the unit disk is $\left(\frac{-1+\sqrt{5}}{2}\right)^{3/2}$ and the absolute minimum of f on the unit disk is $\left(\frac{-1+\sqrt{5}}{2}\right)^{3/2}$. □

3.9. **Vector Fields, Flow Lines, Div, Grad, and Curl.** So:

- A **vector field** on a set U is a map $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$.
- A **flow line** for a given vector field F on U is a path $\gamma : \mathbb{R} \to U$ such that at any point in $\mathbb{R}$,
$$\gamma'(t) = F(\gamma(t)).$$
- For $F : \mathbb{R}^3 \to \mathbb{R}^3$, the divergence of F, denoted $div(F)$ or $\nabla \cdot F$, is given by
$$\left(\frac{\partial F_1}{\partial x}, \frac{\partial F_2}{\partial y}, \frac{\partial F_3}{\partial z}\right)$$
- For $F : \mathbb{R}^3 \to \mathbb{R}^3$, the curl of F, denoted $curl(F)$ or $\nabla \times F$, is given by
$$\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

There are a table of important properties of these operators on page 306 of your text: we reproduce the two most useful here.

Proposition 3.7. *For any C^2 function f, $curl(\nabla f) = 0$ (i.e. the curl of the gradient is 0.) For any C^2 vector field F, $div(curl(F)) = 0$ (i.e. the divergence of the curl is 0.)*