MA1C, WEEKS 6-7: INTEGRALS AND SURFACES

TA: PADRAIC BARTLETT

These notes, like all notes, can be found on my website.

1. The Last Two Week's Work

MT Average: was around 68/90. It was a tough test, so don't feel like you're doomed if you've done poorly; there's still a lot of quarter to go, and definitely room for about anyone (at least in this section) to finish out with a decent grade for the quarter, (if you're willing to work for it.) If you're worried about your performance, come and talk to me and I can try to give you a more detailed picture of what's going on. HW5 Average: 90%; not really any major errors. People seemed to know this stuff.

2. Random Question

Question 2.1. Zombies! (dash a very simple model of how a plague can spread.) *So: take a normal, 8×8 chessboard, and place tokens on some squares of the chessboard. Call these squares "infected." Then, consider the following system designed to simulate how a "plague" might spread on our chessboard:*

- *If at any time, an infected square is touching an uninfected square, put a token on the uninfected square; it is now also infected.*
- *Repeat the process above until there are no more squares which can be infected.*

So: Infecting different squares at the start can yield different final patterns of our chessboard. What is the smallest number of squares it takes to infect the entire board? Can you provide a proof? Can you, furthermore, provide a proof in three words, the second of which is "is?"

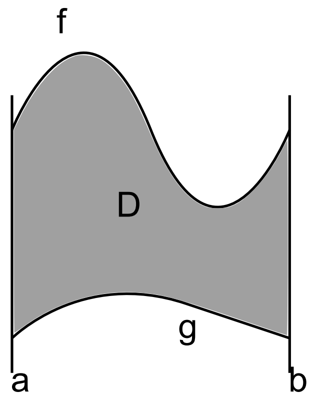
3. Basic Integrals

So: Integrals in multiple variables! i.e.: given a function in multiple variables, we sometimes want to take its **integral** over some region; we furthermore want this integral to kind of capture the same ideas of total value, or average value, that the normal one-dimensional integral did! So: for a function $f : \mathbb{R}^n \to \mathbb{R}$, we denote the integral of f over the n-dimensional box $B = [a_1, b_1] \times [a_2, b_2] \times \ldots [a_n, b_n]$ by $\int_B f ds$, and define this quantity by setting

$$\int_B f ds = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \ldots \int_{a_n}^{b_n} f(x_1, \ldots x_n) dx_1 \ldots dx_n,$$

where each integral above is a simple 1-dimensional integral.

Sometimes, we want to take integrals of shapes that aren't boxes: in the two-dimensional case, we might want to take the integral over a region like the one below, where the top and bottom parts are graphs of some functions – say, in the example below:



So: how do we integrate over this region D? Well – intuitively, if we're integrating over the entire region below, we're just taking the average value of our function with x-values ranging between a and b, and y-values ranging in between $f(x)$ and $g(x)$; so our integral should be

$$\int_D h(x,y)dxdy = \int_a^b \int_{f(x)}^{g(x)} h(x,y)dydx$$

This is in fact what the integral over D is! to see a proof, look in Marsden/Tromba; but it's really just because of our intuitive reasoning.

Similarly, if we had a region E where the y values were bounded within $[a,b]$ and the x values were bounded below and above by two functions $f(y)$ and $g(y)$, respectively, we can define the integral of h over E to be

$$\int_E h(x,y)dxdy = \int_a^b \int_{f(y)}^{g(y)} h(x,y)dxdy.$$

We work an example below, to illustrate how to do these kinds of calculations:

Example 3.1. So: let D be the region given by the bounds $0 \leq x \leq 1, x^2 \leq y \leq x$. What is the integral of the function $f(x,y) = xy$ on this region?

Proof. So: we set up the integral as described above:

$$\begin{aligned}
\int_D f ds &= \int_0^1 \int_{x^2}^x xydydx \\
&= \int_0^1 \frac{xy^2}{2}\Big|_{x^2}^x dx \\
&= \int_0^1 \frac{x^3}{2} - \frac{x^5}{2} dx \\
&= \frac{x^4}{8} - \frac{x^6}{12}\Big|_0^1 = \frac{1}{24}.
\end{aligned}$$

□

4. Change of Variables

So: sometimes, in single-variable calculus, changing variables can make calculations a lot easier to perform. How can we do this in multivariable calculus?

Definition 4.1. so: Suppose that we have a map $T : D \to \mathbb{R}^2$ that's 1-1 and C^1; then, we have the following equality:

$$\int\int_D f(x,y)dxdy = \int\int_{T(D)} f(T_1(x,y),T_2(x,y)) \cdot \begin{vmatrix} \frac{\partial T_1}{\partial x} & \frac{\partial T_1}{\partial y} \\ \frac{\partial T_2}{\partial x} & \frac{\partial T_2}{\partial y} \end{vmatrix} dxdy$$

We call the map T a **change-of-variables** map: common choices of T are the transformation from polar to Cartesian coördinates given by $(r,\theta) \mapsto (r\cos(\theta), r\sin(\theta))$, transformations that shift coördinates over by some fixed quantity or dilate by some fixed amount, or other local homeomorphisms of the plane to itself (reflections, rotations, et.al.)

So: we do an example below, to illustrate how this goes:

Example 4.2. Let

$$f(x,y) = \sqrt{x^2+y^2}$$

and D be the region $[0,1]\times[0,\pi]$. What is $\int\int_D f dxdy$?

Proof. First, notice that the Cartesian to polar coördinate transformation T given by $(r,\theta) \to (r\cos(\theta), r\sin(\theta))$ is 1-1 on D (because sin and cos have periods of 2π and $[0,\pi] \subset [0,2\pi]$), and is C^∞ on D (because we can take arbitrarily many derivatives of the functions $r\cos(\theta), r\sin(\theta)$.) As a result, we can use this transformation T to get that

$$\int_0^1\int_0^\pi \sqrt{x^2+y^2}dxdy = \int_0^\pi\int_0^1 r\cdot rdrd\theta,$$

because the image of the rectangle D under the polar coördinate transformation is the upper half circle described in polar coördinates by $r\in[0,1], \theta\in[0,\pi]$. But this integral is pretty simple to calculate: it's just

$$\int_0^\pi\int_0^1 r^2drd\theta = \int_0^\pi 1/3d\theta = \pi/3.$$

□

5. Path and Line Integrals

For a function $f : \mathbb{R}^3 \to \mathbb{R}$ and a path $c : [a,b] \to \mathbb{R}^3$, we define the **path integral** of f along c to be the integral

$$\int_c f ds := \int_a^b f\circ c(t)\cdot ||c'(t)||dt.$$

More generally, we can extend this notion of a path integral to define the concept of integrating not just a function, but a vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ along a path: we call such integrals **line integrals**, and define them by setting

$$\int_c F\cdot ds := \int_a^b F_1\circ c(t)\cdot c_1'(t)dt + F_2\circ c(t)\cdot c_2'(t)dt + F_3\circ c(t)\cdot c_3'(t)dt,$$

where F_i, c_i are the coördinate functions of the maps F, c.

We do a pair of examples to illustrate the methods here; as these integrals are (given their definitions above) just exercises in integration in straight-up multivariable calculus, there's nothing terribly special about the problems below. So don't worry about trying to find "tricks" or places where you're applying theorems or propositions; this is strictly calculational.

Example 5.1. For $f(x, y, z) = x^{3/2}yz, c(t) = (\frac{t^2}{2}, \frac{t^2}{2}, \frac{t^3}{3})$, c defined for $t \in [0, 1]$, what is $\int_c f ds$?

Proof. So: we calculate.

$$\begin{aligned}
\int_c f &= \int_0^1 f \circ c(t) \cdot ||c'(t)||dt \\
&= \int_0^1 \frac{\sqrt{2}}{24} \cdot t^3 \cdot t^2 \cdot t^3 \cdot \sqrt{(t)^2 + (t)^2 + (t^2)^2}dt \\
&= \int_0^1 \frac{\sqrt{2}}{24} t^9 \cdot \sqrt{2 + t^2}dt \\
(\text{set } u = 2 + t^2) &= \int_0^1 \frac{\sqrt{2}}{48}(u - 2)^4 \cdot \sqrt{u}du \\
&= \frac{\sqrt{2}}{48} \cdot \int_0^1 u^{9/2} - 8u^{7/2} + 24u^{5/2} - 32u^{3/2} + 16u^{1/2}du \\
&= \frac{\sqrt{2}}{48} \cdot (\frac{2}{11} - \frac{16}{9} + \frac{48}{7} - \frac{64}{5} + \frac{32}{3}) \\
&= \frac{5419\sqrt{2}}{83160}.
\end{aligned}$$

□

Example 5.2. So: let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$F(x, y, z) = (x^2, y^2, z^2),$$

and $c : [0, \pi] \to \mathbb{R}^3$ be the path defined by

$$c(t) = (\cos(t), \sin(t), 1).$$

What is

$$\int_c F \cdot ds?$$

Proof. So: by definition, we have

$$\begin{aligned}
\int_c F \cdot ds &= \int_c x^2 \cdot dx + y^2 \cdot dy + z^2 \cdot dz \\
&= \int_0^\pi \cos^2(t) \cdot -\sin(t)dt + \sin^2(t)\cos(t)dt + 1 \cdot 0dt \\
&= \int_0^\pi (\sin^2(t)\cos(t))dt - \int_0^\pi (\cos^2(t)\sin(t))dt \\
&= \int_0^0 u^2du + \int_0^0 u^2du = 0.
\end{aligned}$$

□

6. Parametrized Surfaces

So: um, we should first define what a surface *is.*

Definition 6.1. A **surface** is a collection S of points in $\mathbb{R}^n$ (for some n) such that around any point $s \in S$, there is a small neighborhood of s such that it "locally looks like" $\mathbb{R}^2$; i.e. around any point there is a neighborhood U and a 1-1, onto, and continuous map f from U to the open unit disk $\mathbb{D}$ in $\mathbb{R}^2$. Spheres, tori, and many other shapes are surfaces.

So: given this definition, we can ask when we can define a surface S **universally** with this sort of locally-looks-like-$\mathbb{R}^2$ idea; i.e. whether we can define a function $\Phi : D \subset \mathbb{R}^2 \to S$ that's onto and continuous, for some subset D of the plane. So: if we can do this, we call such a function a **parametrization** of S.

Parametrizations are not always obvious to find, and can require some trickery (i.e. using trig substitutions, polar coördinates, ingenuity) to find. However, there is one case in which a parametrization can always be found: if the surface S is defined as the set of all triples (x,y,z) that satisfy the relation

$$f(x, y) = z$$

for some function f, then we can define a parametrization Φ of S by the map

$$\Phi(x, y) = (x, y, f(x, y)).$$

So: why do we care about parametrizing surfaces? Well, parametrized surfaces are in some sense really easy to understand; they allow us to describe surfaces, even if they live in very high dimensional spaces, with just two coördinates. As well, they allow us to study the surface very easily: given a surface S parametrized by a function $\Phi(u, v)$, we can take the partial derivatives of Φ with respect to u, v at any point to get a pair of tangent vectors at that point. If these two vectors are not collinear, we can take their cross-product to get the normal vector to our surface at this point, which we can use to define the tangent plane at that point. This is really powerful! as really, if we understand the tangent planes of a surface everywhere and its normal vectors everywhere, we really kind of know a lot about the surface.

So: let's work an example.

Example 6.2. Define $\Phi : [0, 2\pi]\times[0, \pi] \to \mathbb{R}^3$ by $\Phi(u, v) = (\cos(u)\cos(v), \cos(u)\sin(v), \sin(u))$. What is the tangent plane to the surface traced by Φ at $(1, 0, 0)$?

Proof. So: we calculate. Note that the point $(1, 0, 0)$ is hit by the value $(0, 0)$.

$$\begin{aligned} T_u &= (-\sin(u)\cos(v), -\sin(u)\sin(v), \cos(u)) \\ T_v &= (-\cos(u)\sin(v), \cos(u)\cos(v), 0) \\ &\Rightarrow T_x \times T_z = \left(-\cos^2(u)\cos(v), -\cos^2(u)\sin(v), -\sin(u)\cos(u)\right) \\ &\Rightarrow T_x \times T_z(0, 0) = (-1, 0, 0) \end{aligned}$$

and thus that the tangent plane to the surface traced by Φ at $(1, 0, 0)$ is the plane given by

$$(x - 1, y - 0, z - 0) \cdot (-1, 0, 0) = 0;$$

i.e. the plane $x = 1$. Which is as expected! as the graph of Φ is the unit sphere. $\square$

So: we can also use parametrizations to calculate areas nicely. Explicitly, for a shape S parametrized by the formula $\Phi : D \to S$, we have the formula

$$Area(S) = \int\int_D ||\Phi_u \times \Phi_v|| dudv.$$

To demonstrate that this works, we calculate the surface area of the unit sphere via our parametrization above:

$$\begin{aligned}
Area(S^2) &= \int\int_D ||\Phi_u \times \Phi_v|| dudv \\
&= \int_0^{2\pi} \int_0^{\pi} \sqrt{(-\cos^2(u)\cos(v))^2 + (-\cos^2(u)\sin(v))^2 + (-\sin(u)\cos(u))^2} dvdu \\
&= \int_0^{2\pi} \int_0^{\pi} \sqrt{\cos^4(u) + \sin^2(u)\cos^2(u)} dvdu \\
&= \int_0^{2\pi} \int_0^{\pi} \sqrt{\cos^2(u)} dvdu \\
&= \int_0^{2\pi} \int_0^{\pi} |\cos(u)| dvdu \\
&= \int_0^{2\pi} \pi \cdot |\cos(u)| du = 4\pi.
\end{aligned}$$

This is indeed the surface area of the sphere! so we can have some faith in this formula.