THINGS ONE CAN DO WITH INTEGRALS! / GREEN'S THEOREM

TA: PADRAIC BARTLETT

1. Random Question

Generalized Tic-Tac-Toe So: contemplate playing tic-tac-toe on a grid that's infinite in all directions, where you're now trying to get 13 in a row as opposed to the standard 3 in a row. Can you come up with a strategy to insure that you never lose?

Question 1.1.

2. Last Week's HW

Average was about 80/90: people did pretty well considering the absolutely epic nature of the set. The only thing that tripped people up was the one "proof" question, that asked you to show that for a curve C decomposed into two paths C_1 and C_2, that $\int_C F = 0$ was equivalent to $\int_{C_1} F = \int_{\bar{C_2}} F$; most people either only showed one side of the inequality, or thought that C_1 and C_2 were actually the same curve but parametrized in different ways (which is really really not true.) If you have any questions/concerns about grading, I'd be happy to address them, as I was the TA responsible for grading this week: shoot me an email at padraic@caltech.edu if this is the case.

3. The Integral of a Function over a Surface

Sometimes, it's useful to be able to study the value of functions over a surface – say, if you have functions which told you information about the surface(local temperature, density, distance from a point). As a result, we want to be able to study the behavior of certain functions on surfaces; specifically, we will often like to be able to study their average behavior, which we can do by defining the notion of an integral over a surface. So, we do this below!

Definition 3.1. For $S \subset \mathbb{R}^3$ a surface parametrized by $\Phi : D \to S$ and $f : S \to \mathbb{R}$ a continuous function, we define the **integral of f over S** to be

$$\int\int_S f(x, y, z)dS = \int\int_D f(\Phi(u, v))||\Phi_u \times \Phi_v||dudv.$$

Here is a (hopefully motivating) example for what we can do with this definition:

Example 3.2. So: Let S be a sphere of radius 1 centered at the origin. What is the x-coördinate of the center of mass of S?

Proof. So: we begin by first noting that the sphere can be parametrized by

$$\Phi(u, v) = (\sin(u) \cos(v), \sin(u) \sin(v), \cos(u)), u \in [0, \pi], v \in [0, 2\pi],$$

and that the function corresponding to the x-coördinate of a surface S is just

$$f(x, y, z) = x. \tag{3.3}$$

So: to find the average x-coördinate, we just need to evaluate the integral

$$\begin{aligned}
\frac{1}{A(S)} \cdot \int\int_S f dS &= \frac{1}{A(S)} \cdot \int_0^{\pi}\int_0^{2\pi} f(\Phi(u, v)) \cdot ||\Phi_u \times \Phi_v|| dvdu \\
&= \int_0^{\pi}\int_0^{2\pi} \sin(u)\cos(v) \cdot \\
&\qquad ||(\cos(u)\cos(v), \cos(u)\sin(v), -\sin(u)) \times (-\sin(u)\sin(v), \sin(u)\cos(v), 0)|| dvdu \\
&= \int_0^{\pi}\int_0^{2\pi} \sin(u)\cos(v) \cdot ||(\sin^2(u)\cos(v), \sin^2(u)\sin(v), \sin(u)\cos(u))|| dvdu \\
&= \int_0^{\pi}\int_0^{2\pi} \sin(u)\cos(v) \cdot \sqrt{\sin^4(u)\cos^2(v) + \sin^4(u)\sin^2(v) + \sin^2(u)\cos^2(u))} dvdu \\
&= \int_0^{\pi}\int_0^{2\pi} \sin(u)\cos(v) \cdot |\cos(u)| dvdu \\
&= 0, \text{b/c } \sin(u)\cos(v) \cdot |\cos(u)| \text{ is periodic with period } 2\pi \text{ in } u.
\end{aligned}$$

We can use similar work to show that the same will hold for the average y and z coördinates – this tells us that the center of mass of our sphere is at the origin! as we hoped. This method, by the way, works for any surface; those of you who will become engineers will do this a lot, I think. □

4. The Integral of a Vector Field over a Surface

So: for similar reasons to the ones stated above, we often will want to take integrals of vector fields over surfaces as well. The following definition shows us how:

Definition 4.1. For $S \subset \mathbb{R}^3$ a surface parametrized by $\Phi : D \to S$ and $F : S \to \mathbb{R}^3$ a continuous vector field, we define the **integral of F over S** to be

$$\int\int_S F(x, y, z) dS = \int\int_D F(\Phi(u, v)) \cdot ||\Phi_u \times \Phi_v|| dudv.$$

To illustrate how we do this, consider the following question:

Example 4.2. So: let S be the section of the monkey saddle $f(x, y) = x^3 - 3xy^2$ with x, y coördinates lying in the box $[-1, 1] \times [-1, 1]$. Let F be the vector field defined by a steady snowfall, $F(x, y, z) = (0, 0, 2)$ (where this is given in inches^3 per hour.) Find the total amount of snow that accumulates on the monkey saddle in a hour.

Proof. So: because the monkey saddle is a surface given by the graph of a function, we have an easy, standard parametrization of it as

$$\Phi(x, y) = (x, y, f(x, y) = x^3 - 3xy^2), (x, y) \in [-1, 1] \times [-1, 1].$$

So: we calculate.

$$\begin{aligned}
\int\int_S F \cdot ds &= \int_{-1}^{1}\int_{-1}^{1} (0,0,2)\cdot(\Phi_x \times \Phi_y)dxdy \\
&= \int_{-1}^{1}\int_{-1}^{1} (0,0,2)\cdot((1,2,3x^2-3y^2)\times(0,1,-6xy))dxdy \\
&= \int_{-1}^{1}\int_{-1}^{1} (0,0,2)\cdot(3y^2-3x^2, 6xy, 1)dxdy \\
&= \int_{-1}^{1}\int_{-1}^{1} 2dxdy \\
&= 8in^3/hr.
\end{aligned}$$

So 8 inches of snow will accumulate on our surface in a hour. Note that this was actually completely independent of the function $f(x, y)$ that we used –i.e. that we would have gotten the same results for any surface that's the graph of $(x, y, f(x, y))$ with $x, y \in [-1, 1]^2$; this is as expected, because the z-coördinate is really irrelevant here! we only care about the xy-dimensions for figuring out how much snow is held here. □

5. Green's Theorem

So, before we can define Green's theorem, we have to first take a little foray into topology:

Definition 5.1. So: we call a path $\gamma : [a, b] \to \mathbb{R}^n$ a **simple closed curve** iff it satisfies the following conditions:

- $\gamma(0) = \gamma(1)$.
- γ is continuous.
- γ is 1-1.

Intuitively: simple closed curves are curves that don't intersect themselves anywhere.

Proposition 5.2. *Suppose that we have a region $D \subset \mathbb{R}^2$ whose boundary can be broken up into simple closed curves $\gamma_1 \ldots \gamma_n$, none of which intersect each other. Then, we can orient the curves γ_i – i.e. define a canonical direction to "walk" about any of the curves γ_i – such that if you were to walk along any such curve in its oriented direction, the region D would always lie on your left-hand-side.*

If you are not persuaded, try drawing some regions that fit the criteria above, and try just orienting them; you'll be surprised.

So: with this machinery defined, we can state Green's theorem:

Definition 5.3. Take a region $D \subset \mathbb{R}^2$ whose boundary can be broken up into simple closed curves $\gamma_1 \ldots \gamma_n$, all of which are oriented as paths γ_i^+ as above, and pick a pair of C^1 functions $P, Q : D \to \mathbb{R}$. Then we have the following identity:

$$\int_{\gamma_1^+ + \ldots \gamma_n^+} Pdx + Qdy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

By the way: this is AMAZINGLY COOL. You're controlling a function's behavior with its derivatives! Completely local data is telling you what's happening on a global scale! It's ridiculously useful.

So: to illustrate this theorem's typical use and power, we work a pair of examples below:

Example 5.4. Show that for any constants α, β and any simple closed curve c^+, that

$$\int_{c^+} \alpha dx + \beta dy = 0.$$

Proof. So: because constant functions are C^∞, if we denote the region encompassed by c^+ by D, we have that

$$\int_{c^+} \alpha dx + \beta dy = \pm \int \int_D (0-0) dx dy = 0.$$

End of proof! □

Example 5.5. Calculate

$$\int_{c^+} (3y^2 + x^3 + y\cos(x))dx + (6xy + y^2 + \sin(x))dy,$$

where c^+ is the curve that traverses a nonagon of unit-length sides in the counter-clockwise direction.

Proof. So: this illustrates one of the key and most frequently used properties of Green's theorem, which is to turn atrocious-looking integrals into nonatrocious things that are often 0. I.e.: note that both $3y^2+x^3+y\cos(x)$ and $6xy+y^2+\sin(x)$ are C^1 because they're composed of polynomials and trig functions, and that c^+ is a SCC encloses a region that Green's theorem applies to: so we can apply Green's theorem to get

$$\int_{c^+} (3y^2+x^3+y\cos(x))dx+(6xy+y^2+\sin(x))dy = \int\int_D (6y+\cos(x))-(6y+\cos(x))dydx = 0.$$

End of proof! □

5.1. **Area.** Another useful application of Green's theorem is to find area: i.e. for a region D with boundary c^+, we have that

$$A(D) = 1/2 \int_c xdy - ydx.$$

An example is calculated below:

Example 5.6. Find the area of a circle with radius r.

Proof. So: the boundary of a circle of radius r is parametrized by the map $c(\theta) = (r\cos(\theta), r\sin(\theta)), \theta \in [0, 2\pi]$; so, by the equation above, we have that

$$A(\mathbb{D}_r) = 1/2 \int_c xdy - ydx = 1/2 \int_0^{2\pi} r^2\cos^2(\theta) + r^2\sin^2(\theta) d\theta = \pi r^2.$$

□

5.2. **The Divergence Theorem.** Another useful form of Green's theorem is the Divergence Theorem, which makes the relation that Green's theorem gives between local and global behavior of a function more obvious: we state it below.

Definition 5.7. For D a region on which Green's theorem holds, ∂D its boundary parametrized in the left-hand-oriented manner by $c^+(t)$, and $n(t)$ the outward-pointing normal vector to ∂D at the point $c^+(t)$, we have that

$$\int_{\partial D} F \cdot nds = \int \int_D div(F) \cdot dA.$$

(Note that the outward-pointing normal vector is given by $((c_2^+)'(t), -(c_1^+)'(t))/||((c_2^+)'(t), -(c_1^+)'(t))||$ for any given parametrization c^+ of the boundary.)

So: this is a useful calculational tool, as we show below:

Example 5.8. For D the unit circle, $F(x, y) = (xy, x + y)$, find $\int_{\partial D} F \cdot n$.

Proof. So: by the divergence theorem, we have

$$\begin{aligned} \int_{\partial D} F \cdot n &= \int \int_{x^2+y^2 \leq 1} (y + 1) dxdy \\ &= \int_0^1 \int_0^{2\pi} (r \cos(\theta) + 1) r dr d\theta \\ &= \int_0^1 2\pi r = \pi r. \end{aligned}$$

□