Math 1c: The Final Review

1 Topics!

The Ma1c final is focused on testing your mastery of the material presented in the second half of this course: in other words, it's a ton of questions about integrals! In the following list, we enumerate the various things we've learned about the integral thus far in the course:

1. **Types of integrals.** We've learned how to take several kinds of integrals in this course:

   - **"Normal" integrals.** Given a region $R \subset \mathbb{R}^n$, we know how to take the integral of any function $F : \mathbb{R}^n \to \mathbb{R}^m$ over such a region by taking **iterated integrals**. For example, if R is some sort of a n-dimensional box $[a_1, b_1] \times \ldots [a_n, b_n]$, we can write $\iint\limits_R F dV$ as the iterated integral
   $$\int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} F dx_n \ldots dx_1.$$

   - **Line integrals.** Given a parametrized curve $\gamma : [a, b] \to \mathbb{R}^n$, we can find the integral of either a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ or a scalar field $g : \mathbb{R}^n \to \mathbb{R}$ along this curve. Specifically, we can express these integrals as the following:
   $$\int_\gamma F \cdot d\gamma = \int_a^b (F \circ \gamma(t)) \cdot (\gamma'(t))dt, \quad \text{and}$$
   $$\int_\gamma g \; d\gamma = \int_a^b (g \circ \gamma(t))||\gamma'(t)||dt.$$

   - **Surface integrals.** Given a parametrized surface S with parametrization $\varphi : R \to S$, $R \subseteq \mathbb{R}^2$, we can find the integral of any function $f : \mathbb{R}^n \to \mathbb{R}$ over S. Specifically, we can express the integral of f over S as the following two-dimensional integral over R:
   $$\int_S f dS = \iint\limits_R (f \circ \varphi(t)) \cdot \left|\left| \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \right|\right| dudv.$$

2. **Tools for evaluating integrals.** Throughout Ma1c, you've ran into many integrals of the above kinds that were difficult or impossible to directly evaluate. Motivated by these problems, we developed a number of theorems and tools about integration, which we repeat here:

   - **Green's theorem.** There are a number of forms of Green's theorem; we state the simpler and most commonly used version here. Suppose that R is a region in $\mathbb{R}^2$ with boundary ∂R given by the simple closed curve C, and suppose that γ is a traversal of C in the counterclockwise direction. Suppose as well that P and Q are a pair of C^1 functions from $\mathbb{R}^2$ to $\mathbb{R}$. Then, we have the following equality:

$$\iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_\gamma \left(Pdx + Qdy\right).$$

- **Stokes' theorem.** Stokes' theorem, quite literally, is Green's theorem for surfaces in $\mathbb{R}^3$ (as opposed to restricting them to lying in the plane $\mathbb{R}^2$.) Specifically, it is the following claim: suppose that S is a surface in $\mathbb{R}^3$ with boundary ∂S given by the simple closed curve C, and suppose that γ is a traversal of C such that the interior of S always lies on the left of γ's forward direction. Suppose as well that F is a C^1 function from $\mathbb{R}^3$ to $\mathbb{R}^3$. Then, we have the following equality:

$$\iint\limits_S (\nabla \times F) \cdot \mathbf{n}\, dS = \int_\gamma F d\gamma.$$

- **Pappus's theorems.** Pappus had a pair of theorems that allowed us to find the surface area and volume of surfaces of revolution with relatively minimal amounts of work. The first, for volume, says the following:
  - Suppose that R is a region in $\mathbb{R}^2$ that doesn't intersect some given axis in $\mathbb{R}^2$, (a, b) is the center of mass of R if we think of it as having uniform density 1, V is the solid of revolution acquired by rotating R about this fixed axis, and h is the distance from (a, b) to this axis, Then, we have that

$$\text{vol}(V) = 2\pi h \cdot \text{area}(R).$$

  The second theorem, for surfaces of revolution, is similar:
  - Suppose that C is a curve in $\mathbb{R}^2$ that doesn't intersect some given axis in $\mathbb{R}^2$, (a, b) is the center of mass of C if we think of it as having uniform density 1, S is the surface of revolution acquired by rotating R about this fixed axis, and h is the distance from (a, b) to this axis, Then, we have that

$$\text{area}(S) = 2\pi h \cdot \text{length}(C).$$

  These two theorems can make calculating volumes and surface areas much more easy than they otherwise might be; see the examples in this handout for some examples where this is true.

- **Change of variables.** A common tactic to make integrals easier is to apply the technique of change of variables, which allows us to describe regions in $\mathbb{R}^n$ using coördinate systems other than the standard Euclidean ones. In general, the change-of-variables theorem says the following:
  - Suppose that R is an open region in $\mathbb{R}^n$, g is a C^1 map $\mathbb{R}^n \to \mathbb{R}^n$ on an open neighborhood of R, and that f is a continuous function on an open neighborhood of the region $g(R)$. Then, we have

$$\int_{g(R)} f(\mathbf{x})dV = \int_R f(g(\mathbf{x})) \cdot \det(D(g(\mathbf{x})))dV.$$

  Specifically, the three most common change-of-variable choices are transitions to the **polar**, **cylindrical**, and **spherical** coördinate systems, which we review here:

- **Polar coördinates.** Suppose that R is a region in $\mathbb{R}^2$ described in polar coördinates: i.e. there is some set $A \subseteq [0, \infty) \times [0, 2\pi)$ such that $\gamma(A) = R$, where γ is the polar coördinates map $(r, \theta) \mapsto (r\cos(\theta), r\sin(\theta))$. Then, for any integrable function $f : \mathbb{R}^2 \to \mathbb{R}$, we have

$$\iiint\limits_{\gamma(A)} f(x, y)dV = \iiint\limits_{A} f(r\cos(\theta), r\sin(\theta)) \cdot r \; dV.$$

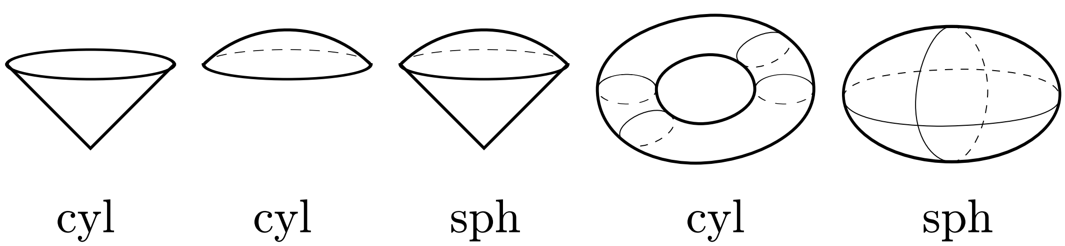
- **Cylindrical coördinates.** Suppose that R is a region in $\mathbb{R}^3$ described in cylindrical coördinates: i.e. there is some set $A \subseteq [0, \infty) \times [0, 2\pi) \times (-\infty, \infty)$ such that $\gamma(A) = R$, where γ is the cylindrical coördinates map $(r, \theta, z) \mapsto (r\cos(\theta), r\sin(\theta), z)$. Then, for any integrable function $f : \mathbb{R}^3 \to \mathbb{R}$, we have

$$\iiint\limits_{\gamma(A)} f(x, y)dV = \iiint\limits_{A} f(r\cos(\theta), r\sin(\theta), z) \cdot r \; dV.$$

- **Spherical coördinates.** Suppose that R is a region in $\mathbb{R}^3$ described in spherical coördinates: i.e. there is some set $A \subseteq [0, \infty) \times [0, 2\pi) \times [0, \pi)$ such that $\gamma(A) = R$, where γ is the spherical coördinates map $(r, \theta, \varphi) \mapsto (r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi))$. Then, for any integrable function $f : \mathbb{R}^3 \to \mathbb{R}$, we have

$$\iiint\limits_{\gamma(A)} f(x, y)dV = \iiint\limits_{A} f(r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi)) \cdot r^2\sin(\varphi) \; dV.$$

One of the trickiest things to do with change of variables is deciding which coördinate system to use on a given set. For example, consider the following five shapes:



To describe the cone, sphere cap, or torus above, cylindrical coördinates are probably going to lead to the easiest calculations. Why is this? Well, all three of these shapes have a large degree of symmetry around their z-axis; therefore, we'd expect it to be relatively easy to describe these shapes as a collection of points (r, θ, z). However, these shapes do *not* have a large degree of rotational symmetry: in other words, if we were to attempt to describe them with the coördinate (r, θ, φ), we really wouldn't know where to begin with the φ coördinate.

However, for the ellipsoid and "ice-cream-cone" section of the ellipsoid, spherical coördinates are much more natural: in these cases, it's fairly easy to describe these sets as collections of points of the form (r, θ, φ).

In general, if you're uncertain which of the two to try, simply pick one and see how the integral goes! If you chose wisely, it should work out; otherwise, you can always just go back and try the other coördinate system.

3. **Applications of the integral.** Finally, it bears noting that we've developed a few applications of the integral to finding volume, surface area, length, and centers of mass. We review these here:

- **Volume, surface area, and length**. If you have a solid V, a surface S, or a curve C, you can find the volume/area/length of your object by integrating the function 1 over that object.
- **Area, via Green's theorem.** If you have a region $R \subset \mathbb{R}^2$ with boundary given by the counterclockwise-oriented curve γ, you can use Green's theorem to find its area as a line integral. Specifically, notice that if $F(x, y) = \left(-\frac{y}{2}, \frac{x}{2}\right)$, we have $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$, and therefore that Green's theorem says that

$$\iint\limits_R 1 \, dA = \int_\gamma \left(-\frac{y}{2}, \frac{x}{2}\right) d\gamma.$$

- **Center of mass.** Suppose that an object A (a curve, surface, or solid) has density function $\delta(\mathbf{x})$. Then, the x_i-coördinate of its center of mass is given by the ratio

$$\frac{\int\limits_A x_i \delta(\mathbf{x}) \, dA}{\int\limits_A \delta(\mathbf{x}) \, dA}.$$

This is pretty much everything we've covered in the second half of our course with respect to the integral! The only other topic we've discussed since the midterm are the operations of **div** and **curl**, which we quickly review here:

1. **Div and curl: definitions.** Given a C^1 vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$, we can defind the **divergence** and **curl** of F as follows:

   - **Gradient**. The **divergence** of F, often denoted either as $\operatorname{div}(F)$ or $\nabla \cdot F$, is the following function $\mathbb{R}^3 \to \mathbb{R}$:

   $$\operatorname{div}(F) = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

   - **Curl.** The **curl** of F, denoted $\operatorname{curl}(F)$ or $\nabla \times F$, is the following map $\mathbb{R}^3 \to \mathbb{R}^3$:

   $$\operatorname{curl}(F) = \nabla \times F = \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right).$$

   Often, the curl is written as the "determinant" of the following matrix:

   $$\det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

2. **Theorems**. We have a pair of rather useful theorems about the divergence and curl of functions, which we state here:

   - For any C^2 function F, $\operatorname{div}(\operatorname{curl}(F))$ is always 0.
   - For any C^2 function F, $\operatorname{curl}(\operatorname{grad}(F))$ is always 0.

   These theorems are a pair of very useful tests that can often tell us that a given function F is not a conservative vector field (i.e. a gradient) or a curl of some other function. For example, if we examined the function $F(x, y, z) = (x, y, z)$, we can immediately tell that F is not the curl of some other function because its divergence is $1 + 1 + 1 \neq 0$.

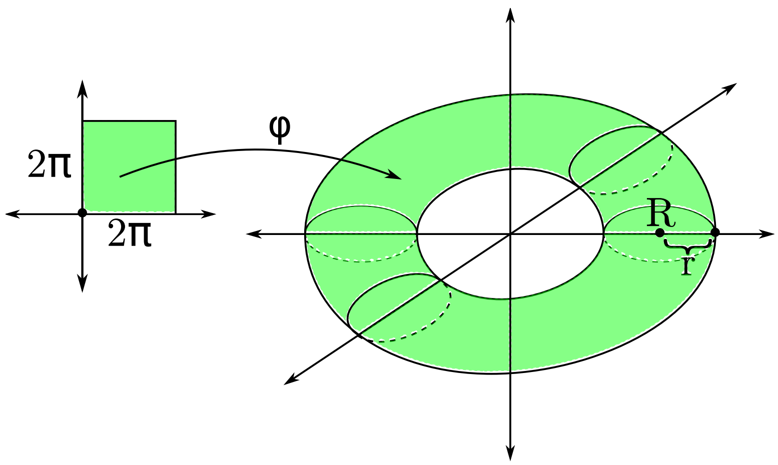
2 Examples!

To illustrate these concepts, we work three examples:

Example. Calculate the surface area of a torus around the circle $x^2 + y^2 = R^2$ with internal radius r.

Solution. From Apostol, or last week's recitation notes, we know that this torus T is parametrized by the map $\varphi : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$, with

$$\varphi(u, v) = (\cos(u)(R + r\cos(v)), \sin(u)(R + r\cos(v)), r\sin(v))\,.$$



One approach to finding the surface area of this torus, then, is just to blindly take the integral of the function 1 over T:

$$\iint_T 1 dT$$

$$= \iint_{[0,2\pi]\times[0,2\pi]} \left|\left|\frac{\partial\varphi}{\partial u} \times \frac{\partial\varphi}{\partial v}\right|\right| \, dudv$$

$$= \iint_{[0,2\pi]\times[0,2\pi]} ||(-\sin(u)(R + r\cos(v)), \cos(u)(R + r\cos(v)), 0) \times (-r\cos(u)\sin(v), -r\sin(u)\sin(v), r\cos(v))|| \; dudv$$

$$= \iint_{[0,2\pi]\times[0,2\pi]} ||(r\cos(u)\cos(v)(R + r\cos(v)), r\sin(u)\cos(v)(R + r\cos(v)), -r\sin(v)(R + r\cos(v)))|| \; dudv$$

$$= \iint_{[0,2\pi]\times[0,2\pi]} \sqrt{(r\cos(u)\cos(v)(R + r\cos(v)))^2 + (r\sin(u)\cos(v)(R + r\cos(v)))^2 + (r\sin(v)(R + r\cos(v)))^2} \; dudv$$

$$= \iint_{[0,2\pi]\times[0,2\pi]} r(R + r\cos(v)) \cdot \sqrt{\cos^2(u)\cos^2(v) + \sin^2(u)\cos^2(v) + \sin^2(v)} \; dudv$$

$$= \iint_{[0,2\pi]\times[0,2\pi]} r(R + r\cos(v)) \cdot \sqrt{\cos^2(v) + \sin^2(v)} \; dudv$$

$$= \iint_{[0,2\pi]\times[0,2\pi]} R \cdot r + r^2\cos(v)) \; dudv = 4\pi^2 R \cdot r.$$

Another approach, which involves much less work, is to apply Pappus's theorem for surface area. Specifically, we can regard our torus as the surface of revolution acquired by revolving the curve $(x-R)^2+z^2=r^2$ around the z-axis. The length of such a circle is $2\pi r$, and the center of mass of a circle is trivially its center, which is at $(R,0,0)$: therefore, the distance of this circle's center of mass from the z-axis is R, and therefore Pappus's theorem says that

$$\text{area}(T) = 2\pi R \cdot 2\pi r = 4\pi^2 R \cdot r.$$

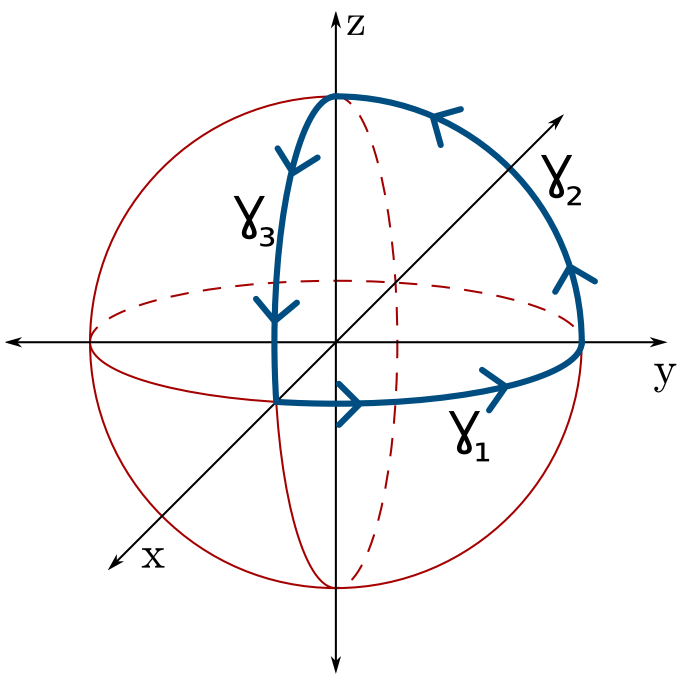
Moral of the story: always check to see if your theorems can save you work!

Example. Let $S = \{(x,y,z) : x^2+y^2+z^2 = 1, x,y,z \geq 0\}$, and $C^+ = \partial S$ be the boundary of S traversed in the counterclockwise direction from high above the z-axis. If $F(x,y,z) = (x^4, y^4, z^4)$, find $\int_C F \cdot dc$.

Solution. One approach we could take here is to parametrize C and simply take line integrals. Specifically: parametrize C as the three curves $\gamma_1, \gamma_2, \gamma_3$, where

$$\begin{aligned}
\gamma_1(t) &= (\cos(t), \sin(t), 0),\\
\gamma_2(t) &= (0, \cos(t), \sin(t)),\\
\gamma_3(t) &= (\sin(t), 0, \cos(t)),
\end{aligned}$$

and t ranges from 0 to $\pi/2$ for each curve.



Then, we'd have that

$$\begin{aligned}
\int_C F dC &= \sum_{i=1}^{3} \int_0^{\pi/2} (F \circ \gamma_i(t)) \cdot (\gamma'(t)) dt \\
&= \sum_{i=1}^{3} \int_0^{\pi/2} -\cos^4(t)\sin(t) + \sin^4(t)\cos(t) dt \\
&= 3\int_0^{\pi/2} -\cos^4(t)\sin(t) + \sin^4(t)\cos(t) dt \\
&= \left(-3\int_0^{\pi/2} \cos^4(t)\sin(t) dt\right) + \left(3\int_0^{\pi/2} \sin^4(t)\cos(t) dt\right)
\end{aligned}$$

To evaluate these last two integrals, use the u-substitution $u = \cos(t)$ on the left and $u = \sin(t)$ on the right:

$$\begin{aligned}
\int_C F dC &= \left(3\int_1^0 u^4 dt\right) + \left(3\int_0^1 u^4 dt\right) \\
&= \left(-3\int_0^1 u^4 dt\right) + \left(3\int_0^1 u^4 dt\right) \\
&= 0.
\end{aligned}$$

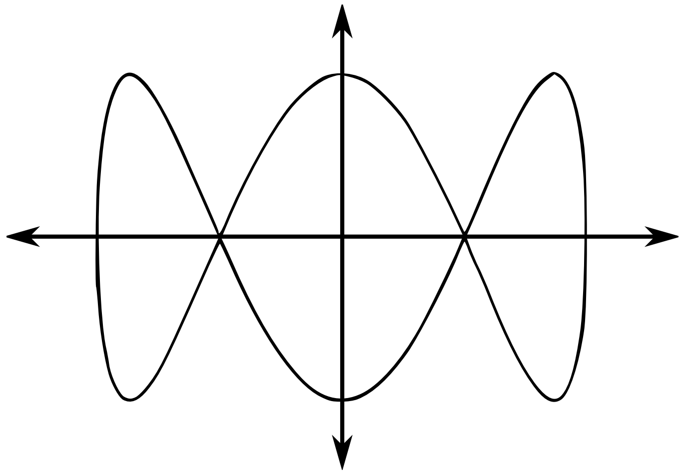
Alternately, for a much faster solution, just use Stokes' theorem, which tells us that the integral of F over C is the integral of $(\nabla \times F) \cdot \mathbf{n}$ over S. Then, because

$$\begin{aligned}
\mathrm{curl}(F) = \nabla \times F &= \left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right) \\
&= (0-0, 0-0, 0-0) \\
&= (0,0,0),
\end{aligned}$$

we know that $(\nabla \times F) \cdot \mathbf{n}$ is identically 0, and thus that the integral of this over S is also zero.

So, again: look for easier solutions, if your initial problem looks thorny!

Example. Find the area of the region R enclosed by the Lissajous curve $\gamma(t) = (\cos(t), \sin(3t))$, where t ranges from 0 to 2π.



Solution. When presented with a region R enclosed by a curve γ, it's really tempting to simply directly apply our Green's theorem for area result, which says that when γ is a simple closed curve oriented counterclockwise, we have

$$\text{area}(R) = \iint\limits_R 1 dA = \int_\gamma \left(-\frac{y}{2}, \frac{x}{2}\right) d\gamma.$$

However, if we just directly apply this here, we'll get that

$$\begin{aligned}\int_\gamma \left(-\frac{y}{2}, \frac{x}{2}\right) d\gamma &= \int_0^{2\pi} \left(-\frac{\sin(3t)}{2}, \frac{\cos(t)}{2}\right) \cdot (-\sin(t), 3\cos(3t))dt \\ &= \frac{1}{2}\int_0^{2\pi} \sin(3t)\sin(t) + \cos(3t)(\cos(t)dt.\end{aligned}$$

By applying your triple-angle formulas $\cos(3t) = 4\cos^3(t) - 3\cos(t), \sin(3t) = 3\sin(t) - 4\sin^3(t)$ (which, by the way, we never expect you to have memorized!), we have that this is

$$\begin{aligned}\int_\gamma \left(-\frac{y}{2}, \frac{x}{2}\right) d\gamma &= \frac{1}{2}\int_0^{2\pi} 3(\sin^2(t) - \cos^2(t)) + 4(\cos^4(t) - \sin^4(t))dt \\ &= \frac{1}{2}\int_0^{2\pi} 3(\sin^2(t) - \cos^2(t)) + 4(\cos^2(t)(1-\sin^2(t)) - \sin^2(t)(1-\cos^2(t)))dt \\ &= \frac{1}{2}\int_0^{2\pi} 3(\sin^2(t) - \cos^2(t)) + 4(\cos^2(t) - \sin^2(t) + \sin^2(t)\cos^2(t) - \sin^2(t)\cos^2(t))dt \\ &= \frac{1}{2}\int_0^{2\pi} 3(\sin^2(t) - \cos^2(t)) + 4(\cos^2(t) - \sin^2(t))dt \\ &= \frac{1}{2}\int_0^{2\pi} \cos^2(t) - \sin^2(t)dt \\ &= \frac{1}{2}\int_0^{2\pi} \cos(2t)dt \\ &= 0.\end{aligned}$$

Um. So, this is clearly false: our curve, by visual inspection, contains more area than 0. What went wrong? Well, our curve γ is **not** a simple closed curve: it has self-intersections!

So: to fix that, we can break up our curve γ into three parts:

- The part where γ's parameter t is restricted to the set $[-\pi/3, \pi/3]$. This is the far-right part of our curve; here, γ is counterclockwise-oriented, and we can thus find the area enclosed by γ by evaluating the integral

$$\frac{1}{2}\int_{-\pi/3}^{\pi/3} \cos(2t)dt = \left.\frac{\sin(2t)}{4}\right|_{-\pi/3}^{\pi/3} = \frac{\sqrt{3}}{4}.$$

- The part where γ's parameter t is restricted to the set $[4\pi/3, 5\pi/3]$. This is the far-left part of our curve; here, γ is also counterclockwise-oriented, and we can thus find the area enclosed by γ by evaluating the integral

$$\frac{1}{2}\int_{4\pi/3}^{5\pi/3} \cos(2t)dt = \left.\frac{\sin(2t)}{4}\right|_{4\pi/3}^{5\pi/3} = \frac{\sqrt{3}}{4}.$$

- The part where γ's parameter t is restricted to the set$[\pi/3, 2\pi/3] \cup [4\pi/3, 5\pi/3]$. Here, γ is clockwise-oriented! Therefore, to find the area enclosed by gamma, we need to take the negative of this signed area, which is

$$\frac{1}{2}\int_{\pi/3}^{2\pi/3} \cos(2t)dt + \frac{1}{2}\int_{4\pi/3}^{5\pi/3} \cos(2t)dt = \ldots = \frac{\sqrt{3}}{2}.$$

  Notice that we've used a curve γ here that was piecewise defined: this is completely OK! The only thing you need to check is that the curve is a simple closed one and counterclockwise-oriented: once you've done that, it can be defined however you like.

Summing these three parts gives us that the area enclosed by our curve is $\sqrt{3}$.