Math 1c TA: Padraic Bartlett

Recitation 1: Open and Closed Sets; Limits and Continuity

Week 1 *Caltech 2011*

1 Random Question

Consider the two possible definitions of "connectedness," for a subset of $\mathbb{R}^n$:

Definition. A subset $X \subset R^n$ is called **path-connected** if for any two points $\mathbf{u}, \mathbf{v} \in X$, there is a path from $\mathbf{u}$ to $\mathbf{v}$: i.e. there is a continuous function $\gamma : [0,1] \to \mathbb{R}^n$ such that $\gamma(0) = \mathbf{u}, \gamma(1) = \mathbf{v}$, and γ's image is contained within X.

Definition. A subset $X \subset R^n$ is called **connected** if there are *no* pairs of open sets U, $V \subseteq R^n$ such that $X = (U \cap X) \cup (V \cap X)$ and both of $U \cap X, V \cap X$ are nonempty.

Are these definitions equivalent? Or is there some set $X \subset R^n$ – say, $\mathbb{R}^2$, if you want to work somewhere specific – that satisfies one of these definitions and not the other?

2 Administrivia

Here are most of the random administrative details for the course:

- My email: padraic@caltech.edu
- My office: 360 Sloan.
- My office hours: 10-11pm on Sunday night and/or by appointment.
- My website: www.its.caltech.edu/∼padraic. Course notes for every recitation will be posted here, ideally within a few days of the recitation.
- HW policy: The course-wide policy is that every student is allowed at most 1 late HW without a note from the deans or health center, with an extension of at most one week. Homeworks after this one will require a note from the health center or the deans: it bears noting that both entities are remarkably kind, and as long as your reason for needing more time is not something like "all-night SC2 marathon," they'll grant an extension.
- Random questions: I post a random question at the start of every recitation! If you've seen the material in rec before, and get distracted, they're meant to offer something mathematically interesting to focus on until the lecture returns to a place you haven't seen. Because we're at Caltech, and pretty much anything we talk about in Math 1 *some* of you have seen before, it struck me as a decent way to avoid boring some students without losing others. If you solve any of them, tell me! I am always interested to see solutions.

3 Open and Closed Subsets of $\mathbb{R}^n$

3.1 Basic definitions.

To refresh your memory, we restate the following definitions:

Definition. In $\mathbb{R}^n$, the **open ball** around $\mathbf{a}$ of radius r, $B_{\mathbf{a}}(r)$, is defined as follows:

$$B_{\mathbf{a}}(r) := \{\mathbf{v} \in \mathbb{R}^n : ||\mathbf{v} - \mathbf{a}|| < r\}.$$

Similarly, the **closed ball** around $\mathbf{a}$ of radius r, $\overline{B}_{\mathbf{a}}(r)$, is defined as follows:

$$\overline{B}_{\mathbf{a}}(r) := \{\mathbf{v} \in \mathbb{R}^n : ||\mathbf{v} - \mathbf{a}|| \leq r\}.$$

Definition. A set $X \subseteq \mathbb{R}^n$ is called **open** if and only if for every point $\mathbf{x} \in X$, there is some radius r such that the open ball $B_{\mathbf{x}}(r)$ is contained entirely within the set X.

A set $X \subseteq \mathbb{R}^n$ is called **closed** if and only if the complement of this set,

$$X^c := \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \notin X\},$$

is open.

Definition. Given a set $X \subseteq \mathbb{R}^n$, we can define the following three objects:

- **Interior**: The interior of X, denoted $\mathring{X}$ or $\text{int}(X)$, is the largest open set contained within X. Equivalently, it can be defined as the subset

$$\mathring{X} := \{\mathbf{x} \in X : \exists r \in \mathbb{R} \text{ s.t. } B_{\mathbf{x}}(r) \subseteq X\}.$$

- **Exterior**: The exterior of X, denoted $\text{ext}(X)$, is the interior of X^c.

- **Boundary**: The boundary of X, denoted $\partial(X)$, is the collection of all points in $\mathbb{R}^n$ that are neither in the interior or exterior of X. Equivalently, it can be defined as the subset

$$\partial(X) := \{\mathbf{v} \in \mathbb{R}^n : \forall r \in \mathbb{R}, (B_{\mathbf{x}}(r) \cap X) \neq \emptyset \text{ and } (B_{\mathbf{x}}(r) \cap X^c) \neq \emptyset\}.$$

Note that, trivially, we have the following proposition:

Proposition 1 *A set X is open iff $X = int(X)$. A set X is closed iff it contains its boundary – i.e. iff $X \supseteq \partial(X)$.*

3.2 Some worked examples.

We work a few examples here, to demonstrate how these definitions get used:

Proposition 2 *The open ball $B_{\mathbf{a}}(r)$ is open, for any $\mathbf{a} \in \mathbb{R}^n$ and any $r > 0$.*

Proof. To show this, all we need to do is pick any $\mathbf{x} \in B_{\mathbf{a}}(r)$, and find a radius r' such that $B_{\mathbf{x}}(r') \subseteq B_{\mathbf{a}}(r)$. How can we do this?

Well: geometrically, what we're looking for is a value of r' such that any point that's within r' of $\mathbf{x}$ will be within r of $\mathbf{a}$. So, if we apply the triangle inequality, we can see that if

$$r' = r - ||\mathbf{x} - \mathbf{a}||,$$

and we take any point $\mathbf{y} \in B_{\mathbf{x}}(r')$, we have that

$$\begin{aligned} ||\mathbf{y} - \mathbf{a}|| &\leq ||\mathbf{x} - \mathbf{a}|| + ||\mathbf{y} - \mathbf{x}|| \\ &< ||\mathbf{x} - \mathbf{a}|| + r - ||\mathbf{x} - \mathbf{a}|| \\ &= r. \end{aligned}$$

Therefore, we have that any point $\mathbf{y}$ in $B_{\mathbf{x}}(r')$ is also in the ball $B_{\mathbf{a}}(r)$. As we've constructed this ball for any point $\mathbf{x}$ in $B_{\mathbf{a}}(r)$, we can conclude that this set is open, by definition.

Question 3 *In $\mathbb{R}$, is the set $[a, b)$ open? Closed?*

Solution. Because any open ball $(b - r, b + r)$ around the real number b contains points both less than and greater than b, it contains both points inside and outside of $[a, b)$ – therefore, b is in the boundary of $[a, b)$. Similarly, by looking at open balls $(a - r, a + r)$ around a, we can argue that a is also on the boundary of $[a, b)$.

What does this mean? Well, we know that any closed set must contain all of its boundary points; thus, because $b \notin [a, b)$, we know this set is not closed. As well, we know that the complement of this set $(-\infty, a) \cup [b, \infty)$ does not contain a, one of its boundary points – therefore, the complement of this set is not closed, and thus $[a, b)$ itself is not open.

Question 4 *In $\mathbb{R}$, is the set $\mathbb{Q}$ open? Closed?*

Solution. First, notice that in any ball $(a - r, a + r)$ around any number a, there are both rational points and irrational points – to see specific examples, simply note that both the sequences $\lfloor a \cdot 10^n \rfloor / 10^n$ and $\lfloor a \cdot 10^n \rfloor / 10^n - \sqrt{2}/n$ converge to a, and that the first of these sequences is made of rational numbers and the second of irrational numbers.

What does this mean? Well, for one, this means that $\mathbb{Q}$ cannot be open – there are no open balls around *any* of its elements contained entirely within $\mathbb{Q}$, as shown above. However, we've also shown that $\mathbb{Q}$ cannot be closed – if we pick any irrational number in $\mathbb{Q}^c$, we've just shown above that in any ball around this irrational there are rational numbers!

Therefore, this set is neither closed nor open.

So: sets can be closed, open, or neither. Can they be both?

As it turns out: yes! If we're discussing subsets of $\mathbb{R}^n$, there are two specific examples: $\emptyset$ and $\mathbb{R}^n$. A proof of this claim is remarkably trivial: to see that $\mathbb{R}^n$, for example, take any point $\curvearrowleft$ in $\mathbb{R}^n$; as the ball $B_{\mathbf{x}}(1)$ is contained within $\mathbb{R}^n$, we've just shown that $\mathbb{R}^n$ is open. To see that it's closed, we just have to show that $\emptyset$ is open.

But this, too, is trivial, as any element of the empty set has an open ball around it contained within the empty set! (Why can we assert this? Well – there are *no* members of the empty set! Therefore, as long as we don't claim that any members of the empty set actually exist, we can ascribe to them whatever properties we would like. I.e.: the statement "every element of the empty set is a purple elephant that commutes with matrix multiplication" is completely true, as there trivially is no counterexample!)

So $\mathbb{R}^n$ is both open and closed; therefore, its complement $\emptyset$, is both closed and open.

4 Limits and Continuity in $\mathbb{R}^n$

4.1 Basic definitions.

We now turn to a discussion of limits and continuity as they exist in $\mathbb{R}^n$; again, we restate a few definitions for your convenience.

Definition. For a set $D \subseteq \mathbb{R}^n$, values $\mathbf{a} \in \mathbb{R}^n, L \in \mathbb{R}^m$, and a function $f : D \to \mathbb{R}^m$, we say that

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$$

if

$$\forall \epsilon > 0 \exists \delta > 0 \forall \mathbf{x} \in D, ((||\mathbf{x} - \mathbf{a}|| < \delta) \to (||f(\mathbf{x}) - L|| < \epsilon)).$$

Notice that this definition is completely identical to the one we used in $\mathbb{R}$, except we've replaced the $|x - a|$ and $|f(x) - L|$'s with $||\mathbf{x} - \mathbf{a}||$ and $||f(\mathbf{x}) - L||$; this is because in $\mathbb{R}^n$, we measure distance using the Euclidean norm $||\cdot||$, which happened to be equal to taking the absolute value $|\cdot|$ when $n = 1$.

Definition. A function $f : D \to \mathbb{R}^n$ is **continuous** at $\mathbf{a} \in D$ iff

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}).$$

Note that this definition is *exactly* the same as it was for $\mathbb{R}$.

4.2 Some worked examples.

Again, to illustrate how these definitions work, we present a few examples:

Question 5 *Consider the function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$, defined as follows:*

$$\begin{aligned} f(x, y) = &\frac{xy}{x^2 + y^2}, \qquad (x, y) \neq (0, 0) \\ &0, \qquad\qquad (x, y) = (0, 0). \end{aligned}$$

Is $f(x, y)$ continuous at $(0, 0)$?

Solution. As it turns out (and as you might be able to guess after attempting to graph or sketch this function), no! This function is not continuous at $(0, 0)$.

How do we prove such a thing? Well, as it turns out, we can use basically the same methods we used to show a function was not continuous in $\mathbb{R}$! One particularly popular method we developed was the following: suppose that we can find a sequence $\{(x_m, y_m)\}_{m=1}^{\infty}$ of elements in $\mathbb{R}^2$ such that

- $\lim_{m\to\infty}(x_m, y_m) = (0, 0)$, and
- $\lim_{m\to\infty} f(x_m, y_m) \neq 0$.

Then, we know that no matter how close we get to $(0, 0)$, for sufficiently large values of m, we'll have (x_m, y_m) is as close as we want to be to $(0, 0)$, and yet $f(x_m, y_m)$ will not be close to 0 – i.e. $\lim_{(x,y)\to(0,0)} f(x, y) \neq 0$! Which is what we want to prove.

So, we just need to find such a sequence. From playing around with this function, it's not too hard to notice that in specific, for any $x \neq 0$,

$$f(x, x) = \frac{x^2}{x^2 + x^2} = \frac{1}{2},$$

and thus that if we let $\{(x_m, y_m)\}_{m=1}^{\infty} = \{(1/m, 1/m)\}_{m=1}^{\infty}$, we have a sequence of points in $\mathbb{R}^2$ that converge to $(0, 0)$ such that $\lim_{m\to\infty} f((1/m, 1/m)) = 1/2 \neq 0$. Thus, our function is not continuous at $(0, 0)$, as claimed.

Let's study something that, at first glance, may look pretty similar:

Question 6 *Consider the function $f(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$, defined as follows:*

$$f(x, y) = \begin{matrix} \frac{xyz}{x^2 + y^2 + z^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y, z) = (0, 0). \end{matrix}$$

Is $f(x, y, z)$ continuous at $(0, 0, 0)$?

Solution. Given the earlier problem, you might expect this function to also be discontinuous at $(0, 0, 0)$; however, after about fifteen minutes of trying to find sequences that converge to any other nonzero value, you might begin to doubt this intuition.

Which, as it turns out, is the correct move – because this function is continuous! Again, to prove that this is continuous, we can use the same methods that we used in $\mathbb{R}$, which were the following:

- Start by taking the quantity $||f(\mathbf{x} - L||$, and try to come up with a simple upper bound on it. In this case, we have for all $(x, y, z) \neq (0,0,0)$,

$$\begin{aligned} ||f(x,y,z) - (0,0,0)|| &= \left| \frac{xyz}{x^2+y^2+z^2} \right| \\ &\leq \left| \frac{\max\{|x|^3, |y|^3, |z|^3\}}{x^2+y^2+z^2} \right| \\ &\leq \left| \frac{\max\{|x|^3, |y|^3, |z|^3\}}{\max\{x^2, y^2, z^2\}} \right| \\ &= |\max\{|x|, |y|, |z|\}|. \end{aligned}$$

(This trick, where we bounded a polynomial expression xyz from above by assuming all of your variables $x, y, z \ldots$ were just the largest one $\max\{|x|, |y|, |z|\}$, and bounded another polynomial $x^2 + y^2 + z^2$ from below by only taking the largest monomial $\max\{|x|^2, |y|^2, |z|^2\}$ – this is *super super useful*! Do this.)

- Now, we want to bound the $||\mathbf{x} - \mathbf{a}||$ portion of our proof from below, so that it is related to the simple upper bound we just got. In this case, we can use the observation that

$$\begin{aligned} ||(x,y,z) - (0,0,0)|| &= \sqrt{x^2+y^2+z^2} \\ &\geq \sqrt{\max\{x^2, y^2, z^2\}} \\ &= |\max\{|x|, |y|, |z|\}|. \end{aligned}$$

- Now, given any $\epsilon > 0$, use this knowledge to pick a value of $\delta > 0$ such that whenever $||\mathbf{x} - \mathbf{a}|| < \delta$, $||f(\mathbf{x} - L|| < \epsilon$! In particular, for our example, we've shown the following:

$$\begin{gathered} ||f(x,y,z) - (0,0,0)|| \leq |\max\{|x|, |y|, |z|\}|, \text{ and} \\ |\max\{|x|, |y|, |z|\}| \leq ||(x,y,z) - (0,0,0)||. \end{gathered}$$

Therefore, if we pick $\delta = \epsilon$, we will have

$$||f(x,y,z) - (0,0,0)|| \leq |\max\{|x|, |y|, |z|\}| \leq ||(x,y,z) - (0,0,0)|| < \delta = \epsilon,$$

which concludes our proof. So we're done!

4.3 Epilogue.

A fun "bonus" random question: Suppose you consider the generalization

$$f(x_1, \ldots x_n) = \begin{cases} \dfrac{x_1 \cdot \ldots \cdot x_n}{x_1^m + \ldots x_n^m}, & \mathbf{x} \neq \mathbf{0}, \\ 0, & \mathbf{x} = \mathbf{0}. \end{cases}$$

of the function we've worked with in the last two questions. What values of n and m make it continuous at $\mathbf{0}$?